Multivariable Calculus

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$

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Contents

About this course

This is a course in Calculus of several variables, also known as Multivariable Calculus The main topics of the course are:

1: Parameterizing curves and surfaces

2: Linear approximations to curves and surfaces

3: Partial derivatives

4: Optimization in multiple dimensions

5: Multivariable Taylor series expansions

6: Integrals in multiple dimensions

7: Differential Forms and Stokes' Theorem

My lecture notes will be provided for the class for the class for those who wish to concentrate on lecture or get ahead in the reading to make understanding easier. You can assist me and the class by letting me know when you see any mistakes or typos in the notes, and I will correct them as quickly as possible. Although class attendance is not required, it is certainly recommended!

How can I get an A in this course?

1: Come to every class on time (try to come five minutes early if you are habitually late).

2: Turn off your phone/laptop/other external communication device when in lecture, they suck your attention away even when you are not looking at them.

3: Read all the homework questions and try them by yourself first (I'd schedule an hour on Wednesday afternoon or evening to do this), then talk to others or me if you have problems.

4: Do actually talk to me (or friends in the class) though if you can't do a homework problem.

5: When I hand out the study guides, learn the definitions presented.

6: Do extra practice on problem types that you find difficult. [There's a wealth of examples on the web, please ask me if you need any assistance in finding problems and we can go Google hunting together.]

Levels of Mathematics

Mathematics is about patterns of relationships between abstract objects. Something that is abstract could be something as simple as the number 2, or as complex as Brownian motion. Roughly speaking, mathematics is about understanding the permissible ways that you can transform problems and ideas to make them easier to work with. One way to think about it is that there are four levels of mathematics.

1: Techniques.

2: Ideas and concepts.

3: Rigorous proof.

4: Automatical proof verification.

The first level is all about using theorems and facts to solve real problems. For instance, in high school you learned an algorithm for how to multiply two numbers written in decimal format. At the second level, we try to understand mathematical facts through a different prism. For instance, why does a times b equal b times a for numbers a and b? Well, a times b is the area of a rectangle that has length a on the horizontal side and b on the vertical side. Whereas b times a is the area of the rectangle with length b on the vertical side and a on the horizontal side. But we got that rectangle just by rotating the first rectangle 90 degrees, which does not change the area.

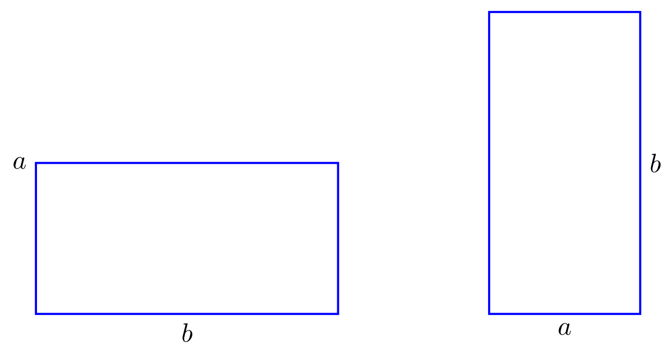


Figure 1: $a \cdot b = b \cdot a$, a geometric perspective.

At the third level, we have the notion of rigorous proof. At this level it is important to define exactly what a particular mathematical object is, so that we can derive true facts that follow logically from the definitions. For instance, for defining $a \cdot b$ where a and b are positive integers, we could build a Turing machine that when fed a tape of a and b in unary, returns $a \cdot b$. This would give a precise definition, and so then it would be possible to logically derive (prove) the result that $a \cdot b = b \cdot a$. This is the level at which most mathematics is done in journals.

At the fourth level, everything has entirely been reduced to symbols, and the permissible steps have been written out precisely enough that it is possible for a computer to check the proof. Very few mathematicians work at this level, but it is the most precise level and the least likely to contain errors.

About this course This text contains a complete semester course in Multivariable Caclulus. It starts at the beginning, building from the ground up using set-theory. There are a lot of definitions in the text about derivatives, integrals, and the linear algebra that we need to deal with them properly.

Intuition, Definitions, Facts, Lemmas, and Theorems

In this text you will encounter various boxes, containing intuitions, definitions, facts, lemmas, and theorems.

Intuition This is a way about thinking about a mathematical object that is not a formal definition, but gives the idea behind the object. For instance, saying that a derivative is the slope of the tangent line to the function doesn't give you a way to calculate it, but gives a visual idea of what the derivative is all about.

Definition This is the formal mathematical definition of an object. Often these definitions are in terms of mathematical objects defined earlier. For instance, the formal definition of a derivative of a function f with one real input and output is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

In other words, the definition of derivative is in terms of an earlier idea of limits.

There is no need to prove that definitions are true, they are true by definition.

Facts Once we have definitions, we can use them to prove things that are true about mathematical objects using logic. For instance, given the definition of derivative above,

$$\begin{aligned}
[x^2]' &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} 2x + h \\
&= 2x.
\end{aligned}$$

The fact that the derivative of x^2 is $2x$ follows logically from the definition, whereas the definition iteself is just taken to be true.

Lemmas These are facts that are important, but not generally as important as Theorems.

Theorems These are the most important facts in the course. They get used a lot in many different situations, and so it is good to know these by heart.

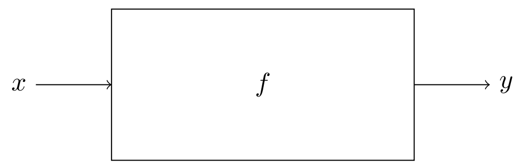
Review: Single Variables Calculus

In single variable calculus, you studied many things, but the essential concepts were the following:

- Sequences
- Limits
- Series
- Derivatives
- Integrals

0.1 Intuitive notions of Calculus

In single variable Calculus, limits, derivatives, and integrals are applied in the context of functions that had a single input, and a single output. A function is a computation that links two variables together. So given the input variable, you can calculate the output variable. For instance, f in the following picture links x and y together (here $y = f(x)$.)



Intuition 1
A **function** $y = f(x)$ is a description of how to calculate the value of the output variable y given the input variable x.

We will write $f : \mathbb{R} \to \mathbb{R}$ to mean that $y = f(x)$ where both x and y are real numbers.

Intuition 2
A **sequence** is an unbounded stream of numbers such that for any integer n, we can calculate the nth number in the sequence.

For instance, $a_i = i^2$ is a sequence. We could also have written $a(i) = i^2$, or just write out the terms of the sequence for $i = 1, 2, \ldots$:

$$1, 4, 9, \ldots.$$

As n goes to infinity, the numbers $a_i = i^2$ get bigger and bigger, and so we say the limit is ∞. The numbers $b_i = 1/i^2$ get smaller and smaller as $i \to \infty$, so we say $\{b_i\}$ has limit 0.

Intuition 3
The **limit** of a sequence is the number that the terms of the sequence get close to as we move farther and farther out in the sequence. If for any fixed number, the sequence is eventually larger than that number forever, then the limit is ∞.

If we add the first n terms in a sequence together, we can a new sequence. For instance, if the original sequence was $a_i = i^2$, the new sequence is

$$s_1 = a_1 = 1, s_2 = a_1 + a_2 = 5, \ s_3 = a_1 + a_2 + a_3 = 14, \ldots$$

Intuition 4
If the limit of the summed sequence is finite as the number of terms added goes to infinity, call the limit of the sum a **series**.

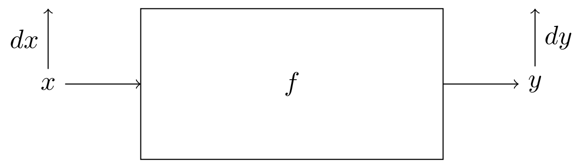
For instance,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

is a series.

The key to Calculus is understanding the notion of the differential, which is a tiny change in a variable value. For a variable, the differential is the variable name with a d in front of it. So a tiny change in variable y is dy, a tiny change in variable x is dx, a tiny change in variable r is dr, and so on.

Now, when we move the input x a little bit, then the output y changes a little bit. In the picture, this looks like:



For many functions, the amount that y changes (dy) is proportional to the amount that x changes (dx). We call that constant of proportionality, the derivative.

Intuition 5
The **derivative** of the function $y = f(x)$ is the instaneous rate of change of y in terms of x, or $f'(x) = dy/dx$.

Next up is the notion of an integral. An integral is a sum of an infinite number of infinitely small things.

Intuition 6
Suppose that some quantity (profit, area, probability) is accumulating at rate $r(t)$ which changes for $t \in [a, b]$. Then the **integral** of $r(t)$ over $t \in [a, b]$ is the total amount of accumulation that occurs. We write

$$\int_{t \in [a,b]} r(t)\ dt.$$

Every integral consists of three parts: the limits of the integral, the integrand (the rate funtion) and the differential.

For example:

- Oil is being pumped out of the ground at rate $f(t)$. How much total oil is drawn for t from t_0 to t_1?

$$\int_{t_0}^{t_1} f(t)\ dt.$$

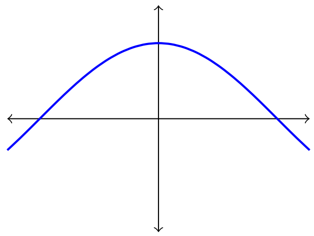
- For two functions $f(x)$ and $g(x)$, area between the function accumulates at rate $|f(x) - g(x)|$. What is the area between the functions for x from 0 to 1?

$$\int_{x \in [0,1]} |f(x) - g(x)|\ dx.$$

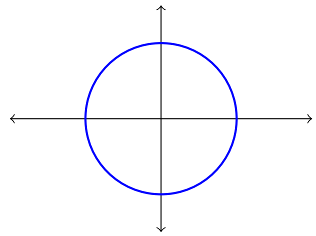
0.2 Formal notions of Calculus

In order to prove facts, lemmas, and theorems about mathematical objects, we need precise definitions of those objects. Those precise definitions are contained in this section. Start with the most important part of mathematics: functions.

Definition 1
A **function** is a collection of points (a, b) such that for each value a there is at most one b such that (a, b) is in the set of points.



A function



Not a function

Next comes sequences.

Definition 2
A **sequence** is a function that takes as input a positive or nonnegative integer.

Now the limit of a sequence is a number such that for any difference away from that number, eventually the tail of sequence lies in that difference.

Definition 3
Say that $\lim_{n\to\infty} a_n = L$ if for all $\epsilon > 0$, there exists an n such that $\{a_n, a_{n+1}, a_{n+2}, \ldots\} \subseteq [L-\epsilon, L+\epsilon]$.

Then a series is just the limit of the partial sums of a sequence.

Definition 4
Suppose that the sequence $\{a_i\}$ satisfies

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i = L.$$

Then call L a **series**, and write

$$\sum_{i=1}^{\infty} a_i = L.$$

Next comes a limit of a function. There are various ways to define a limit, this way comes from

Definition 5
For function $f : \mathbb{R} \to \mathbb{R}$, write

$$\lim_{x\to a} f(x) = L$$

if for all $\epsilon > 0$, there exists a $\delta > 0$, so that for all x satisfying $|x-a| \leq \delta$, it holds that $|f(x) - L| \leq \epsilon$.

Now that we have a limit, we can formally define a derivative.

Definition 6
A **derivative** of a function $y = f(x)$ where x and y are real numbers is

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

when this limit exists.

Finally, we tackle integrals. The first kind of integral we define is when the rate function only changes values a finite number of items in the interval.

Definition 7
A function $f : [a, b] \to \mathbb{R}$ is **simple** if it only changes value at a finite number of points in $[a, b]$. That is, there exist $a = a_0 < a_1 < a_2 < \cdots < a_n = b$ such that f is constant over (a_i, a_{i+1}). The **integral** of a simple function is

$$\int_{x \in [a,b]} f(x)\ dx = \sum_{i=1}^{n} f((a_i + a_{i+1})/2)(a_{i+1} - a_i).$$

Now for non-simple functions.

Definition 8
Let $f : [a, b] \to \mathbb{R}$ be a function such that for each i, $g_i \le f \le h_i$, where $\{g_i\}$ and $\{h_i\}$ are simple functions. If

$$\lim_{i \to \infty} \int_{x \in [a,b]} g_i(x)\ dx = \lim_{i \to \infty} \int_{x \in [a,b]} h_i(x)\ dx = I,$$

then call I the **integral** of f over $[a, b]$.

This definition is interesting because it is not entirely clear at first glance that there couldn't be more than one value of I that equals the limits of different sequences. In fact, it turns out that if $\lim \int g_i = \lim \int h_i$, then no matter what $g_i \le f \le h_i$ were chosen the common limit has to be the same value.

1 Introduction to the course

Question of the Day Suppose an airplane at time t is at position $(10\cos(t), 10\sin(t), t)$. How far does it travel in the time from $t = 10$ to $t = 20$?

Today

- What's in the course?
- Parameterizing curves
- Understanding differentials

Calculus of several variables

- Specifically, calculus of functions of several variables
- So far, most of what you've done has been in one dimension
- 2 dimensional space: Google maps, images, (height,weight)
- 3 dimensional space: Needed for positions (one more dimensional for time)
- U.S. production of: steel, cars, oil, lumber, pharmaceuticals, electricity,...

3 main concepts

1: Differentials

- Small changes in a variable
- dx = small change in x
- $d\theta$ = small change in θ

2: Linear approximations of a function

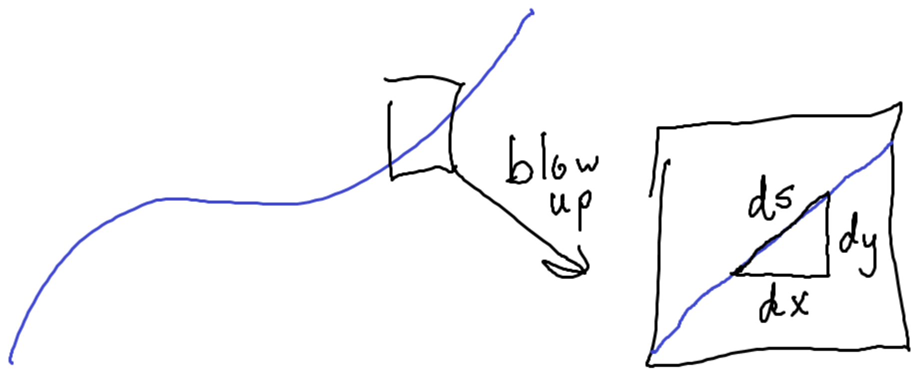
- Curves look like lines when you zoom in on them
- Linear functions easier to work with, so approximation helpful in many ways
- In one dimension, slope of linear approximation called a *derivative*
- Expressed in differentials, $y = f(x)$, $f'(x) = dy/dx$.
- In higher dimensions, derivatives related to best linear approximation

3: Integrals

- Summing up terms that involve a differential.
- Finding area under a curve/probability of event/total accumulation of flow at a rate
- In higher dimensions: finding volume under a surface/finding length of curves

Example: Question of the day

- Zoom in on a curve:



- Pythagorean Theorem:

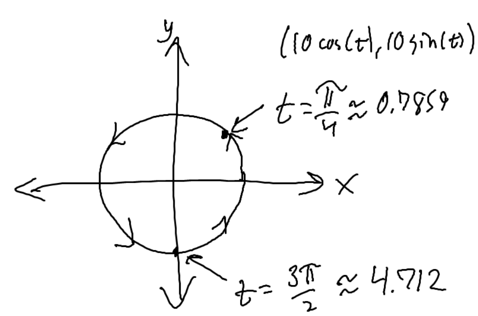
$$ds^2 = dx^2 + dy^2$$
$$ds = \sqrt{dx^2 + dy^2}$$

- Note: Pythag. works in higher dimensions too!

$$ds^2 = dx^2 + dy^2 + dz^2$$
$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

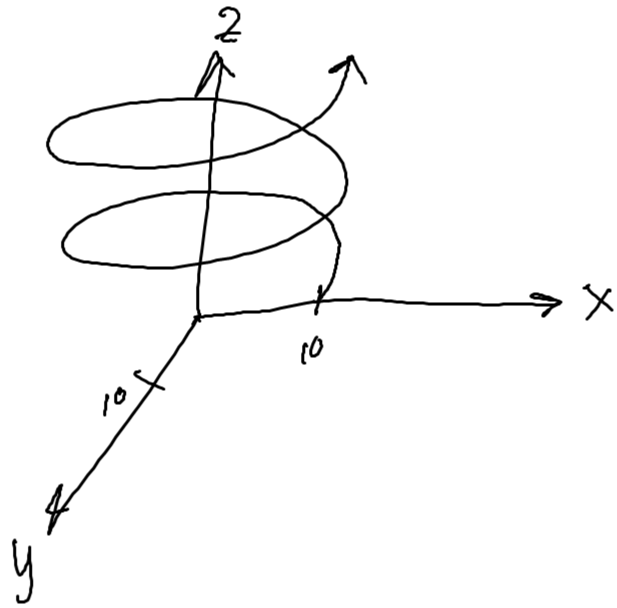
Parameterizing curves

- How do we find dx, dy, dz?
- First step: write x, y, and z in terms of t
- Qotd: $(x(t), y(t), z(t)) = (10\cos(t), 10\sin(t), t)$
- What does this look like?



- Then add in the z part and it is rising upwards

- Isometric viewpoint:



$$x'(t) = \frac{dx}{dt}, \quad y'(t) = \frac{dy}{dt}, \qquad z'(t) = \frac{dz}{dt}$$

Multiplying by dt:

$$x'(t)\ dt = dx, \quad y'(t)\ dt = dy, \qquad z'(t)\ dt = dz$$

So that gives:

$$\begin{aligned} ds &= \sqrt{(x'(t)\ dt)^2 + (y'(t)\ dt)^2 + (z'(t)\ dt)^2} \\ &= \sqrt{x'(t)^2 dt^2 + y'(t)^2 dt^2 + z'(t)^2 dt^2} \\ &= |dt|\sqrt{x'(t)^2 + y'(t)^2 + z'(t)} \end{aligned}$$

- Total length $= \int_{t\in[10,20]} ds = \int_{t\in[10,20]} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\ dt$
- Note: $|dt| = dt$ here since limits go from 10 to 20. For $a < b$

$$\int_a^b f(x)\ dx = \int_b^a f(x)(-dx) = -\int_b^a f(x)\ dx.$$

In general, have the following formula

Definition 9
Let $(x_1(t), \ldots, x_n(t))$ be a parameterized curve where each $x_i(t) \in C^1$. Then the **differential arc length** of the curve for $t \in [a, b]$ is

$$ds = \sqrt{x_1'(t)^2 + \cdots + x_n'(t)^2}\ dt.$$

Adding up all the small pieces (differentials) of a variable is what we call integration.

Fact 1
Integrals obey

$$\int_{t=t_0}^{t_1} ds = s(t_1) - s(t_0).$$

Adding up all the little tiny differential arc length gives the total length of the curve.

Definition 10
The **arc length** of a parameterized curve is

$$S = \int_{t=a}^{b} ds = \int_{t=a}^{b} \sqrt{x_1'(t)^2 + \cdots + x_n'(t)^2}\ dt.$$

- We used the calculation using Pythagorean theorem on differentials to motivate the definition, but this is a definition, not a lemma or theorem.
- This result transforms one differential to another. Another such transformation result that you should be familiar with is the Fundamental Theorem of Calculus. For $y = f(x) \in C^1$

$$dy = f'(x)\ dx.$$

- Integrals of continuous function always exist
- Need derivatives of x_i to exist and be continuous

Definition 11
Say that $f \in C^i$ if the ith derivative of f is continuous.

Solving the Qotd

-

$$x(t) = 10\cos(t), \quad y(t) = 10\sin(t), \quad z(t) = t,$$
$$x'(t) = -10\sin(t), \quad y'(t) = 10\cos(t), \quad z'(t) = 1,$$

So

$$s = \int_{10}^{20} \sqrt{10^2 \sin^2(t) + 10^2\cos^2(t) + 1^2}\ dt$$
$$= \int_{10}^{20} \sqrt{101}\ dt = 10\sqrt{101} \approx 100.4$$

(to 4 sig figs)

Problems

1.1: Show that $(0, 2)$ and $(3, 0)$ are perpendicular vectors

1.2: Show that $(2, 3)$ and $(-6, 4)$ are perpendicular vectors.

1.3: Set up the integral to find the arclength of along the following curves:

(a) $P(t) = (\cos(t), \sin(t))$, $0 \leq t \leq \tau$
(b) $P(t) = (t, t^2, t^3)$, $0 \leq t \leq 1$
(c) $P(t) = (\exp(t), t, \sqrt{t})$, $1 \leq t \leq 2$
(d) $P(t) = (1, 1/(1 + t))$, $0 \leq t \leq 10$

1.4: Use Wolfram Alpha to numerically solve the above integrals to 4 significant figures.

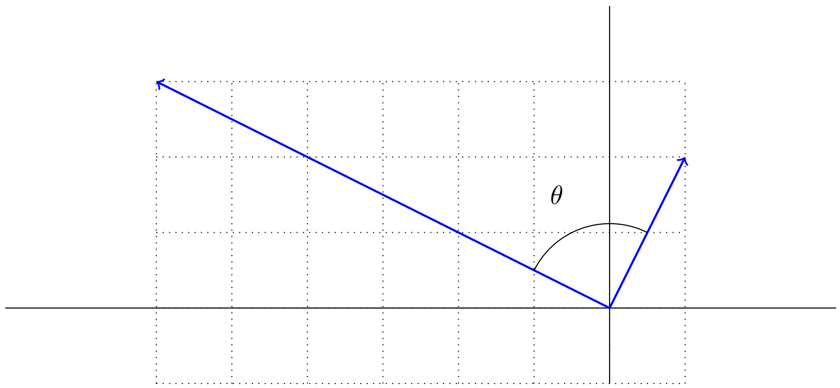
2 Vectors, sets, and the distance formula

Question of the Day Two dune buggies leave location $(0,0)$, One heads for $(1,2)$, the other for $(-6,3)$. What is the angle between them?

Today

- Determining when vectors are perpendicular
- The norm of a vector

Picture



Qotd: What is θ?

Looks like $\theta = 90° = \pi/2$. But is it?

2.1 Vectors

- Vectors are anything that can be added together or scaled.
- Examples
  - Functions with one real input and one real output: x, x^2 are functions, can be scaled to give $3x$ and $-2x^2$ which are also functions, which can then be added to give $3x - 2x^2$ which is also a function
  - Random variables: X and Y are rolls of a fair six sided die, $3X - 2Y$ is also a random variable
  - Sequences: $\{a_i\} = 1, 2, 3, 4, \ldots$ and $\{b_i\} = 1, 4, 9, 16, \ldots$ are sequences, so is $1, -2, -9, \ldots = 3\{a_i\} - 2\{b_i\}$
  - Points in space: $a = (1, 2)$, $b = (-6, 3)$, then $3a - 2b = (-3, -3)$.
- Unless otherwise specified, vectors in this course refer to points in space.

2.2 Sets

Definition 12
A **set** is an unordered collection of objects. The objects in the set are called **elements** of the set. The order of elements in the set does not matter.

Notation

- Curly braces $\{$ and $\}$ indicate a set.
- {green,red,blue} indicates green, red, and blue are the elements of the set.
- Names of sets usually capital letters. For example, $A = \{1, 2, 3\}$ is a set of numbers.
- Write $1 \in A$ to indicate that 1 is an element of A.
- A **space** is a set that has extra structure.
- For example, the set of integers is a space that has addition and multiplication

Notation for points in space

- Often use Blackboard boldface for spaces
- Examples
  - $\mathbb{R}$ = the set of real numbers
  - $\mathbb{Z}$ = the set of integers
  - $\mathbb{Q}$ = the set of rational numbers
- A point in 2-D space is represented by two real numbers

Definition 13
For sets A and B, the set of 2-tuples $\{(x, y) : x \in A, y \in B\}$ is denoted by the **Cartesian product** or **direct product** $A \times B$.

- Example: $(4, -2) \in \{-2, 0, 2, 4\} \times \{-2, 0, 2\}$, but $(4, 4)$ is not.
- Informally, functions take input values in one set A and return an output vlaue in another set B
- Functions are computational rules
- Formally, functions are subsets of points in the Cartesian product

Definition 14
A **function** f from A to B (write $f : A \to B$) is a collection of points in $A \times B$ such that if (a, b) and (a, c) are both in $A \times B$, then $b = c$.

Universal Quantifiers and set notation

- $\forall$: "for all" or "for every"
  - $(\forall x > 3)(2x > 6)$ is true because no matter what value of x greater than 3 I choose, $2x$ will be greater than 6.
  - $(\forall x > 3)(2x > 10)$ is false becuase if $x = 3.5$, then $2x = 7$ which is not greater than 10.
- $\exists$: "there exists"
  - $(\exists x > 3)(2x > 6)$ is true. For instance, if $x = 5$ then $2x = 10$. There only has to be at least one value of x that makes it true for the statement to be true.
  - $(\exists x > 3)(2x > 10)$ is true
  - $(\exists x > 3)(2x < 1)$ is false
- Can combine them:

- $(\forall y)(\exists x)(x + y > 6)$ is true. No matter what value of y you pick, you can choose x (for instance $x = 7 - y$ works) so that $x + y > 6$.
- $(\exists y)(\forall x)(x + y > 6)$ is false

- $:$ = "such that"
  - $(\forall x : |x| > 3)(x^2 > 9)$

Definition 15
For sets $A_1, \ldots, A_n$ the set of n-tuples $\{(x_1, \ldots, x_n) : (\forall i)(x_i \in A_i)\}$ is denoted by the **Cartesian product** $A_1 \times A_2 \times \cdots \times A_n$. When $A_1 = A_2 = \cdots = A_n = A$, write $A_1 \times \cdots \times A_n = A^n$

- In other words, a function is a set of input values in A and output values in B such that every unique input is associated with a unique output.

Real vectors

- Ex: $(3, 2) \in \mathbb{R}^2$, $(1, 2, -3) \in \mathbb{R}^3$
- If $\vec{v} \in \mathbb{R}^n$, say that $\vec{v}$ is a real vector in n-dimensional space. Usually write just v rather than $\vec{v}$
- Vectors can be added together

Definition 16
For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, then $a + b = (a_1 + b_1, \ldots, a_n + b_n)$.

Example: $(1, 2) + (-6, 3) = (1 - 6, 2 + 3) = (-5, 5)$

Vectors can be scaled:

Definition 17
For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, $\alpha a = (\alpha a_1, \ldots, \alpha a_n)$.

Example: $2 \cdot (1, 2) = (2 \cdot 1, 2 \cdot 2) = (2, 4)$.

Definition 18
A **vector space** consists of a set of vectors V and a set of scalars S such that

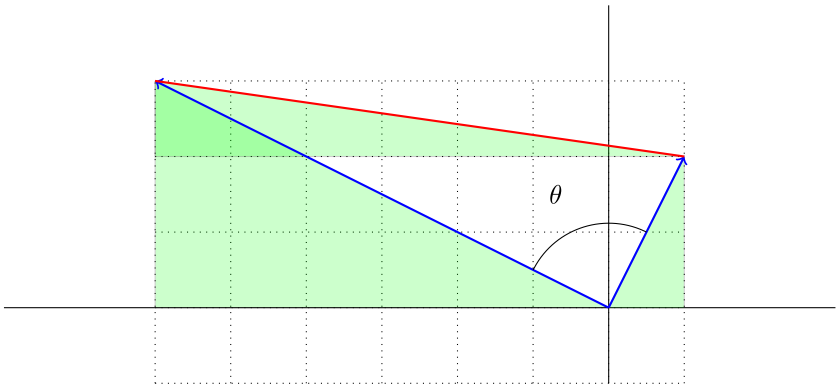
$$(\forall s_1, s_2 \in S)(\forall v_1, v_2 \in v)(s_1 v_1 + s_2 v_2 \in V).$$

Say that vectors spaces are *closed* under scalar multiplication and vector addition.

Qotd: does $\theta = 90°$?

Theorem 1 (Pythagorean Theorem)
Let A, B, and C be the points of a triangle (so they are not colinear). Let α be the length of side AB, β the length of side BC, and γ the length of side AC. Then $\alpha^2 + \beta^2 = \gamma^2$ if and only if the angle between AC and BC is 90 degrees.

- Often used as: if have a right angle, then $\alpha^2 + \beta^2 = \gamma^2$.



Qotd: What is θ?

- For instance, length from $(0,0)$ to $(-6,3)$ is

$$\sqrt{(-6-0)^2+(3-0)^2}=\sqrt{45}$$

  Length from $(0,0)$ to $(1,2)$:

$$\sqrt{(1-0)^2+(2-0)^2}=\sqrt{5}$$

  Length from $(1,2)$ to $(-6,3)$:

$$\sqrt{(1-(-6))^2+(2-3)^2}=\sqrt{50}.$$

- Since

$$(\sqrt{45})^2+(\sqrt{5})^2=(\sqrt{50})^2,$$

  the angle is $\boxed{\theta=\pi/2\approx 1.570}$.

- In general, the length of a vector is called the Euclidean norm of that vector

Definition 19
The **Euclidean norm** of $v=(v_1,\ldots,v_n)$, written $\|v\|$, equals

$$\sqrt{v_1^2+v_2^2+\cdots+v_n^2}.$$

This is also known as the **magnitude** of the vector.

Problems

2.1: What is $(x,y)+(3,4)$ written as a single vector?

2.2: True or false: the vectors $(1,2,3)$ and $(3,2,1)$ are the same vector.

2.3: The Euclidean norm of $(-5,0)$ is what?

2.4: List the points in $\{2,3\}\times\{-1,0,1\}$.

2.5: Write the following sums of scaled vectors as a single vector:

(a) $(2,3)+(-1,4)$
(b) $(x,y)+(w,z)$
(c) $(2,3)+2(-1,4)$
(d) $(x,y)+2(w,z)$

2.6: Find $\|(3,-2,0,2)\|$.

3 Angles between vectors

Question of the Day What is the angle between $(1,2)$ and $(-1,3)$?

Today

- Scaling vectors
- Angles between vectors
- Orthogonal/perpendicular angles in $\mathbb{R}^n$
- Correlation

Using norms

- Gave a way to check if the angle between two vectors is a right angle
- Distance formula gives a way to find lengths/Euclidean norm/L_2 norm $\|v\| = \|v\|_2$
- Taxicab, or L_1 norm: $\|v\|_1 = |v_1| + \cdots + |v_n|$
- L_∞ norm, $\|v\| = \max\{|v_1|, |v_2|, \ldots, |v_n|\}$

3.1 Scaling vectors

Scaling vectors

- One way to change a vector is to scale it (also known as multiplication)
- For $v = (v_1, \ldots, v_n)$ $\alpha v = (\alpha v_1, \ldots, \alpha v_n)$
- How does that change the norm?

$$\begin{aligned}\|\alpha v\| &= \sqrt{(\alpha v_1)^2 + \cdots + (\alpha v_n)^2} \\ &= \sqrt{\alpha^2 v_1^2 + \cdots + \alpha^2 v_n^2} \\ &= \sqrt{\alpha^2 (v_1^2 + \cdots + v_n^2)} \\ &= |\alpha| \sqrt{v_1^2 + \cdots + v_n^2} \\ &= |\alpha| \, \|v\|\end{aligned}$$

- Note that if v is scaled by the inverse of its norm, the new norm is 1:

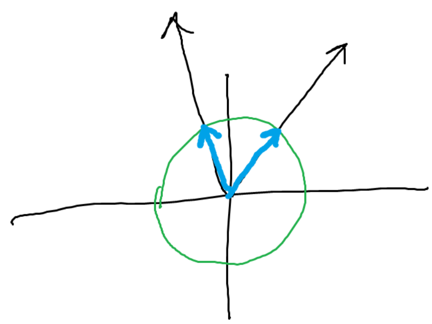
$$\left\| \frac{v}{\|v\|} \right\| = \frac{\|v\|}{\|v\|} = 1$$

Definition 20
If v is a nonzero vector, then $v/\|v\|$ is the **normalization** of v.

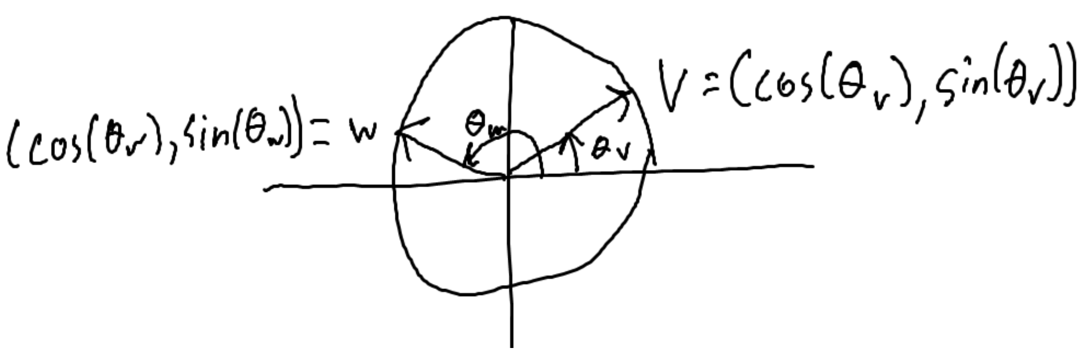
3.2 Angles between vectors

Why this is useful in finding angles:

- Scaling v and w does not change the angle between them!



- So you might as well normalize them
- $\text{angle}(v, w) = \text{angle}(v/\|v\|), \text{angle}(w/\|w\|)$
- Assume without loss of generality that $v = (v_1, v_2)$ and $w = (w_1, w_2)$ have norm 1
- Then they lie on the unit circle: $v_1^2 + v_2^2 = 1$, $w_1^2 + w_2^2 = 1$, so the picture looks like this:



Now something interesting happens. Consider $v_1w_1 + v_2w_2$:

$$\begin{aligned}\cos(\theta_v)\cos(\theta_w) + \sin(\theta_V)\sin(\theta_w) =& \\ \cos(-\theta_v)\cos(\theta_w) - \sin(-\theta_V)\sin(\theta_w) &= \cos(\theta_w - \theta_v),\end{aligned}$$

but $\theta_w - \theta_v$ is the angle between w and v!

- More generally,...

Definition 21
The **dot product** or **inner product** between two vectors $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ is

$$v \cdot w = v_1w_1 + \cdots + v_nw_n.$$

With this definition, it is easy to show the following.

Fact 2
For $\alpha, \beta \in \mathbb{R}$ and $v, w \in V$, $(\alpha v) \cdot (\beta w) = (\alpha\beta)(v \cdot w)$.

This motivates the following definition:

Definition 22
For any $v, w \in \mathbb{R}^n$, the angle θ between v and w can be found using

$$\cos(\theta) = \frac{v \cdot w}{\|v\| \cdot \|w\|}.$$

Qotd :

$$\cos(\theta) = \frac{(1,2)\cdot(-1,3)}{\|(1,2)\|\,\|(-1,3)\|} = \frac{1(-1)+2(3)}{\sqrt{5}\sqrt{10}} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

That makes $\boxed{\theta = \pi/4 \approx 0.7853}$

3.3 Norm of a vector

Definition 23
For vector space V, say that $\|\cdot\| : V \to \mathbb{R}^{\geq 0}$ is a **norm** if the following three properties hold:

1: $(\forall \alpha \in \mathbb{R})(\forall v \in V)(\|\alpha v\| = |\alpha|\,\|v\|)$

2: $(\forall v, w \in V)(\|v + w\| \leq \|v\| + \|w\|)$ (triangle inequality)

3: $(\forall v)(\|v\| = 0 \to v = 0)$

This gives rise to the following notion of orthogonality (which means same as perpendicular in $\mathbb{R}^n$)

Definition 24
Say that v and w are **orthogonal** or **perpendicular** if $v \cdot w = 0$.

3.4 Angles between random variables

- Points in space are vectors
- Random variables in probability are also vectors
- Ex: $X \sim \mathsf{Unif}\{1,2,3,4,5,6\}$ if $\mathbb{P}(X=1) = \mathbb{P}(X=2) = \cdots = \mathbb{P}(X=6) = 1/6$.
- The average value of a random variable is the sum of the values it takes on times the probability it takes on those values.

$$\mathbb{E}[X] = (1/6)(1) + (1/6)(2) + \cdots + (1/6)(6) = 3.5$$
$$\mathbb{E}[X^2] = (1/6)(1^2) + (1/6)(2^2) + \cdots + (1/6)(6^2) = \frac{91}{6} = 15.1666\ldots$$

- Then the inner product between X and Y is defined to be:

$$\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

In probability this has a special name, it is the **covariance** between X and Y.

- The norm of X is called the **standard deviation**, and is

$$\mathrm{SD}(X) = \sqrt{\mathbb{E}[X^2] - \mathbb{E}[X]^2}$$

- The cosine of the angle between X and Y is then

$$\frac{\mathrm{Cov}(X, Y)}{\mathrm{SD}(X)\,\mathrm{SD}(Y)}$$

and in probability this is called the **correlation**. Because it is the cosine of an angle, it must lie between 0 and 1.

- Orthogonal random variables X and Y are called **uncorrelated**.

Problems

3.1: What is $(x, y) \cdot (3, 4)$?

3.2: What is $(3)(3, -2)$?

3.3: What is $c(3, -2)$ where $c \in \mathbb{R}$?

3.4: What is the cosine of the angle between $(1, 0)$ and $(0, 1)$?

3.5: If v and w are perpendicular, what is $v \cdot w$?

3.6: (a) What is $(2, 3) \cdot (-1, -1)$?

(b) What is $(1, 0, -1) \cdot (7, 3, 4)$?

(c) What is $(x, y) \cdot (2, -2)$?

3.7: Find the angle between vectors $(2, 3)$ and $(-1, -4)$.

4 Differentiation of Curves

Question of the Day At time t, an airplane is at position $(10\cos(t), 10\sin(t), t)$. What is the velocity at time t?

Today

- Linear approximation to parameterized curves

Some physics:

- The derivative of position is velocity
  - Velocity is itself a vector with direction and magnitude
- The derivative of velocity is acceleration

Parameterized curves

- Parameter is a name for the common input to several functions
- Notation: $x \mapsto x^2$ (read as: x maps to x^2) means $f(x) = x^2$.
- $P(t) = (x(t), y(t), z(t))$ is a curve parameterized with t

Definition 25
A parameterized curve $P(t) = (v_1(t), \ldots, v_n(t))$ is **continuous** if each $v_i(t)$ is continuous.

Derivatives

- Recall the definition of a derivative (:= means defined as)

$$f'(t) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

  Remember that the slope of a line is rise over run. Here the rise is the change in the output value: $f(x+h) - f(x)$. The run is the change in the input value: $(x+h) - x = h$.)

- Try that with the parameterized curve:

$$\begin{aligned} P'(t) &= \lim_{h \to 0} \frac{P(t+h) - P(t)}{h} \\ &= \lim_{h \to 0} \frac{(x(t+h), y(t+h), z(t+h)) - (x(t), y(t), z(t))}{h} \\ &= \lim_{h \to 0} \left(\frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h}, \frac{z(t+h) - z(t)}{h} \right) \\ &= (x'(t), y'(t), z'(t)) \end{aligned}$$

- Rule easy: differentiate vectors component wise!

Qotd

$$\begin{aligned} P(t) &= (10\cos(t), 10\sin(t), t) && \text{position} \\ P'(t) &= (-10\sin(t), 10\cos(t), 1) && \text{velocity} \\ P''(t) &= (-10\cos(t), -10\sin(t), 0) && \text{acceleration} \end{aligned}$$

More physics

- Speed is the magnitude of velocity $= \|P'(t)\|$.
- Distance traveled = speed · time traveled.
- Differential distance traveled is $ds = \|P'(t)\| \ dt$, where

$$\|P'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}.$$

- Same answer we got for differential arc length earlier!

4.1 Rules for differentation

Fact 3
Let $f, g, \in C^1$. Then

1: $\dfrac{d}{dt}(f+g) = \dfrac{df}{dt} + \dfrac{dg}{dt}$

2: For all $c \in \mathbb{R}$, $\dfrac{d(cf)}{dt} = c\dfrac{df}{dt}$

3: $\dfrac{d(fg)}{dt} = fg' + f'g$

- These first two properties make differentiation of curves a *linear operator*. (Note that operator is yet another term for function.

Definition 26
An operator $\mathcal{L} : A \to B$ where A and B are vector spaces is a **linear operator** if

$$(\forall a, b \in \mathbb{R})(\forall x, y \in A)(\mathcal{L}(ax + by) = a\mathcal{L}(x) + b\mathcal{L}(y)).$$

- Any time you can pull out constants and split up additions, you have a linear operator.
- Example: differentiation

$$[3x + 2x^2]' = 3[x]' + 2[x^2]'$$

- Example: integration

$$\int_0^2 3x + 2x^2 \ dx = 3\int_0^2 x \ dx + 2\int_0^2 x^2 \ dx.$$

- Example: multiplying real numbers by 4

$$4(3x + 2y) = 12x + 8y = 3(4x) + 2(4y).$$

Definition 27
A parameterized curve $P(t) = (v_1(t), \ldots, v_n(t))$ is in class C^i if the ith derivative of $P(t)$ exists and is continuous.

Fact 4
Let $P, Q : \mathbb{R} \to \mathbb{R}^n$ be in C^1. Then

1: $\dfrac{d}{dt}(P+Q) = \dfrac{dP}{dt} + \dfrac{dQ}{dt}$

2: For all $c \in \mathbb{R}$, $\dfrac{d(cP)}{dt} = c\dfrac{dP}{dt}$

3: $\dfrac{d(P \cdot Q)}{dt} = P \cdot Q' + P' \cdot Q$

- Note P, P', Q, Q' are all vectors, so $\cdot$ means dot product here.

Example:

- What is the derivative of $\|X(t)\|^2 = X(t) \cdot X(t)$?
- Use the product rule:

$$\begin{aligned}\frac{d(X(t) \cdot X(t))}{dt} &= [X(t)]' \cdot X(t) + X(t) \cdot X'(t) \\ &= 2X'(t) \cdot X(t).\end{aligned}$$

- Physics interpretation: $X(t)$ is velocity, $X'(t)$ is acceleration, $\|X(t)\|$ is speed.
- Speed is constant iff speed squared is constant.
(iff equals if and only if)
- Speed squared is constant iff $[\|X(t)\|^2]' = 0$
- $[\|X(t)\|^2]' = 0$ iff $2X'(t) \cdot X(t) = 0$ iff $X'(t)$ orthogonal to $X(t)$
- Example: rock tied to a string swinging around your head. The acceleration is perpendicular to the direction of motion, so the speed of the rock does not change.
- One more rule, for product of function times parameterized curve...

Fact 5 (Mixed product rule)
Say $f(t) : \mathbb{R} \to \mathbb{R}$ and $P(t) : \mathbb{R}^n \to \mathbb{R}$ are in C^1, then

$$\frac{d(f(t)P(t))}{dt} = f(t)P'(t) + f'(t)P(t).$$

- Nothing new to remember, same as normal product rule!
- Example:

$$\begin{aligned}\frac{d}{dt}(P + e^{2t}Q) &= P' + \frac{d(e^{2t}Q)}{dt} \\ &= P' + e^{2t}Q' + 2e^{2t}Q.\end{aligned}$$

4.2 Linear approximation of curves

Recall linear approximation of function:

$$f(x) \approx f(x_0) + (x - x_0)f'(x_0).$$

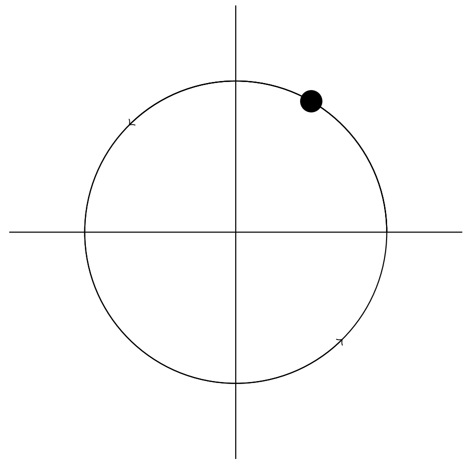
- RHS (right hand side) is called the *tangent line* to the curve.

Same idea for paramerized curves

Definition 28
The **tangent line** to a parameterized curve $P(t)$ at $P(t_0)$ is

$$P_1(t) = P(t_0) + (t - t_0)P'(t_0)$$

Example: For $P(t) = (\cos(t), \sin(t))$, what is the tangent line at $t_0 = \pi/3$?



- $P(t) = (\cos(t), \sin(t))$, so $P'(t) = (-\sin(t), \cos(t))$
  $P(\pi/3) = (1/2, \sqrt{3}/2)$, $P'(\pi/3) = (-\sqrt{3}/2, 1/2)$
  So the answer is:

$$\begin{aligned} \ell(t) &= (1/2, \sqrt{3}/2) + (t - \pi/3)(-\sqrt{3}/2, 1/2) \\ &= \boxed{(1/2 + \pi\sqrt{3}/6 - t\sqrt{3}/2, \sqrt{3}/2 - \pi/6 + t/2)} \end{aligned}$$

- Check in WA

```
parametric plot (cos(t),sin(t)) and (1/2+pi*sqrt(3)/6-t*sqrt(3)/2,
sqrt(3)/2-pi/6+t/2) for t from -2pi to 2pi
```

Problems

4.1: True or false: Speed at a point is always a real number.

4.2: True or false: Let $f(x, y) = 2x + xy^2$. Then $f \in C^1$.

4.3: State whether or not the following parameterized curves are in C^1.

(a) $P(t) = (t, t^2, e^t)$
(b) $P(t) = (|t|, \sin(t))$

4.4: A particle moves along a trajectory so that at time t its location is $(t, t^2, \exp(t))$.

(a) What is its velocity at time $t = 1$?
(b) What is its acceleration at time $t = 1$?
(c) What is its speed at time $t = 1$?

4.5: For $P(t) = (\sin(t), t^2)$, find the tangent line to P at $t = 0$.

4.6: Suppose that a particle has a circular path parameterized by

$$C(t) = ((3\sqrt{3}/2)\sin(t), ((3\sqrt{3}/2)\cos(t)).$$

(a) Find the velocity of the particle at $t = \tau/4$.

(b) Find the speed of the particle at $t = \tau/4$.

(c) Find the acceleration of the particle at $t = \tau/4$.

(d) Write the equation of the tangent line to the path at $t = \tau/4$.

5 Level sets and partial derivatives

Question of the Day How can higher dimensional functions be visualized?

Today

- Types of functions
- Multiple inputs, multiple outputs
- Level sets
- Partial derivatives

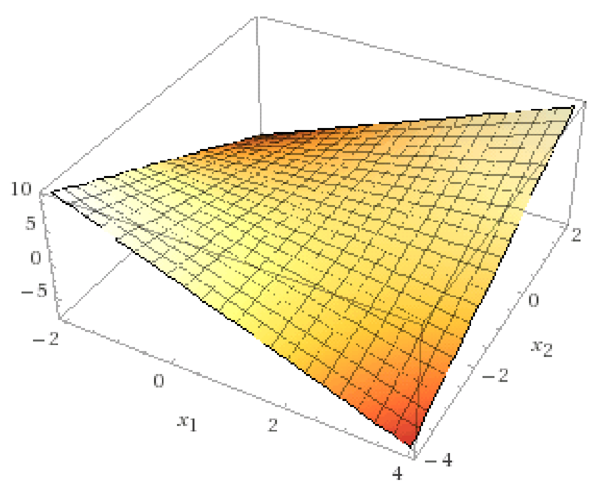
Function idea A *function* takes one or more inputs, does some computation, and then returns one or more outputs. Depending on how many inputs (and how many outputs, these can be described in various ways.

- Curves in the plane: $P : \mathbb{R}^1 \to \mathbb{R}^2$
  Input is time, output is a point in the 2D plane
- Curves in space: $P : \mathbb{R}^1 \to \mathbb{R}^3$
  Input is time, output is a point in 3D space
- Surface over Plane: $f : \mathbb{R}^2 \to \mathbb{R}^1$
  Input is point in 2D plane, output is the height above the plane
  Example: Topographical map giving altitude at every location Example functions:

$$f(x, y) = x^2 + y^2$$
$$f(a, b) = ab + \cos(a)$$
$$f(x_1, x_2) = x_1 - x_2 + x_1 x_2$$

- Real-valued function: $f : \mathbb{R}^n \to \mathbb{R}$
- Sketching the graph can be difficult, in WA:

```
plot x_1 - x_2 + x_1 x_2 for x_1 from 0 to 1 and x_2 from 0 to 1
```

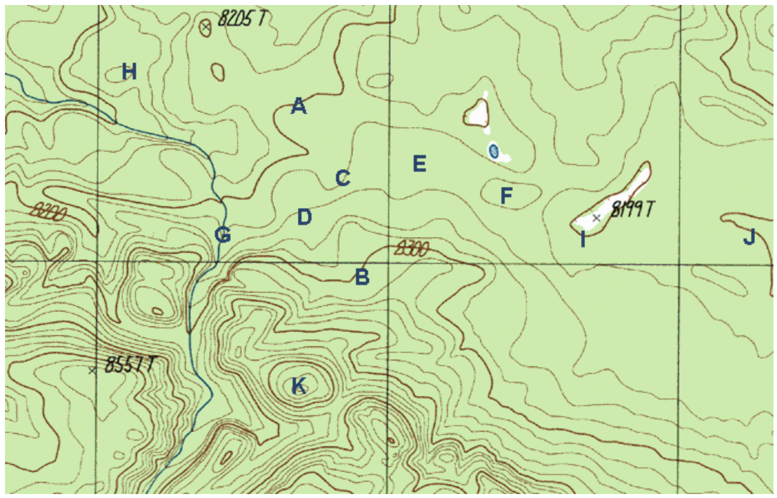


- Visualization even tougher in 3D

$$f(x, y, z) = x^2 + y^2 + z^2$$
$$f(x_1, x_2, x_3) = x_1 - x_2 + 3x_3$$

Level curves

- Topographical map draws lines where the elevation is equal
- Ex: Connect all points at 8300 feet

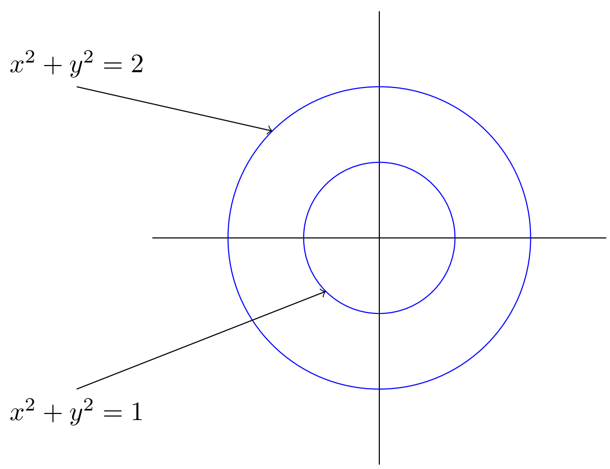


- Lines form distorted circles around peaks
- In mathematics, these lines are called level sets.

Definition 29
The set of points for which $f(x,y) = c$ is called the **level curve** of f at c.

Example

$$f(x,y) = x^2 + y^2$$
$$\text{Level curve} = \{(x,y) : x^2 + y^2 = c\}.$$



- All the level curves of this function are concentric circles.
- When $f : \mathbb{R}^3 \to \mathbb{R}$, then level sets are level surfaces.

Definition 30
The set of points for which $f(x,y,z) = c$ is called the **level surface** of f at c.

5.1 Partial derivatives

- In this subsection, assume f only has one output, so $f : \mathbb{R}^n \to \mathbb{R}$.
- A function $f : \mathbb{R}^2 \to \mathbb{R}$ can be viewed as a height map.
- If at location (x, y), could try climbing in x direction, or y direction.
- How steep is the mountain in these directions?
- First: need to know that can climb a little bit in any direction.

When all of the elements of A are also in B, then we say that A is a subset of B

Definition 31
The set A is a **subset** of B (write $A \subseteq B$) if

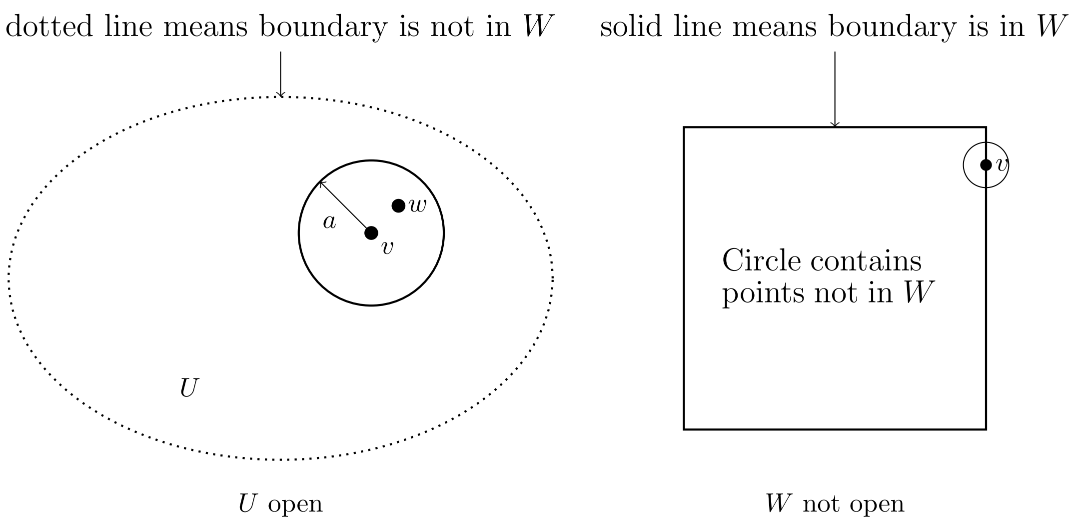
$$(\forall a \in A)(a \in B).$$

Recall: $\forall$ = for all, $\exists$ = there exists

Definition 32
A set $U \subseteq \mathbb{R}^n$ is an **open set** if

$$(\forall v \in U)(\exists a > 0)(\forall w : \|v - w\| < a)(w \in U)$$

- In words: a set U is open if for all points v in U, there is a measure of closeness a such that if a point w is within distance a of v, then w is also in U.
- Picture:



Definition 33
Let e_i be the vector that is 1 is coordinate i, and is 0 elsewhere. Let $f : U \to \mathbb{R}$, where $U \in \mathbb{R}^n$ is an open set, and $x \in U$. Then the **partial derivative** of f with respect to x_i is

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x + h \cdot e_i) - f(x)}{h}$$

when this limit exists.

- The e_i vector is used to ensure that we are looking in the ith coordinate value.
- The ∂ symbol can be read as "dell" or "partial".

Practical partial derivatives

- No one uses limit definition to find derivatives
- No one uses limit definition to find partial derivatives
- To find partial derivative with respect to x_i, treat all other variables as constants.
- Example: find $\partial(xy^2)/\partial x$.
  - Treat y^2 as a constant, $\partial(xy^2)/\partial x = y^2$.
- Example: find $\partial(xy^2)/\partial y$.
  - Treat x as a constant, $\partial(xy^2)/\partial y = 2xy$.

The gradient of a function collects all the partial derivatives together into a vector.

Definition 34
The **gradient** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is

$$\text{grad}(f) = \nabla f := \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

- Find the gradient of $f(a, b, c, d) = ab + 2cd$
- Answer: $\boxed{\nabla f = (b, a, 2d, 2c)}$

Problems

5.1: Graph the level sets of $x = (1/2)y^2$.

5.2: Find the following partial derivatives.

(a) $\partial(x^2 y)/\partial x$.
(b) $\partial(x^2 y)/\partial y$.
(c) $\partial(x^2 y)/\partial z$.
(d) $\partial(\exp(-2x))/\partial x$.
(e) $\partial(r/w)/\partial r$.

5.3: Find the following partial derivatives.

(a) $\partial[x^2 y + 2y]/\partial y$.
(b) $\partial[x^2 y + 2y]/\partial x$.
(c) $\partial[x^2 y + 2y]/\partial z$.

6 Proofs and limits

Question of the Day Show that $\lim_{(x,y)\to(1,1)} 1 + x + y = 3$ using the definition of limit.

Today

- Limits of real-valued functions
- Introduction to logic and proofs

Definition 35
A function $f : A \to B$ is **real-valued** if $B \subseteq \mathbb{R}$.

- Real valued functions can be optimized, and might have limits.
- What does $\lim_{x\to a} f(x) = L$ mean? Assume $f : \mathbb{R}^n \to \mathbb{R}$ is real-valued, and $x, a \in \mathbb{R}^n$.
- It means that as x gets close to a, then $f(x)$ gets close to L.
- The distance from x to a is $\|x - a\|$. The distance from $f(x)$ to L is $|f(x) - L|$.
- More precisely, $\lim_{x\in a} f(x) = L$ means that we can play the following game. I specify how close I want $f(x)$ to L, then can you can find a distance such that for x within your distance of a, $f(x)$ is within that distance of L.
- Use ϵ for my distance (how close $f(x)$ is to L) and δ for your distance (how close x is to a).
- Formal definition (note $\Leftrightarrow$ means if and only if)

Definition 36
The **limit** as x approaches a of $f(x)$ equals L means

$$\lim_{x\to a} f(x) = L \Leftrightarrow (\forall\epsilon > 0)(\exists\delta)(\forall x : \|x - a\| < \delta)(|f(x) - L| \leq \epsilon).$$

- How can I use this definition to do proofs?
- For instance, how do I prove the qotd?
- Start simpler: Prove $(\forall\epsilon > 0)(2\epsilon > 0)$.
  - When you see a "forall" symbol, you have to *instantiate the quantifier*. This means that we assume ϵ is an arbitrary number that is at least 0.

    Proof Let $\epsilon > 0$.

  - Then you can use algebra to get to the final answer. In this case, multiplying by 2 gets to the final step, so the complete proof is:

    Proof Let $\epsilon > 0$.
    Then $2\epsilon > 0$ (by multiplying both sides by 2). $\square$.

  - The symbol $\square$ is one way to indicate that a proof is finished.
  - Also can write *Q.E.D.* for *quad erat demonstratum*.

Working with $\exists$

- *You* get to pick anything you want as long as it works.
- Ex: Prove $(\exists x)(2x > 15)$.
  - The last line of the proof will be: "So $2x > 15$. $\square$"
  - The first line of the proof will be: Let $x =$, and I get to choose what x equals.

    Proof Let $x =$
    Then $2x > 15$.
  - Since I get to pick x, I will pick $x = 10$. Then $2x = 20$, and I'm done. Here's the final proof:

    Proof Let $x = 10$
    Then $2x = 20 > 15$. $\square$

Combining $\forall$ and $\exists$

- Here's where it gets interesting, when you combine for all and there exists.
- Example: prove $(\forall x)(\exists y)(x + y > 10)$
  - The overall framework of the proof will look like this:

    Proof Let $x \in \mathbb{R}$
    Let $y =$
    [Stuff happens]
    Then $x + y > 10$. $\square$
  - Note that I cannot pick a single y that works for all x. For instance, if $y = 10$, but $x = -2$, then $x + y = 8 \leq 10$. So y must depend on x.
  - Time for some side work: if $x+y > 10$, then $y > -x+10$. What is a number bigger than $-x+10$? How about $-x + 11$? Try this out!

    Proof Let $x \in \mathbb{R}$
    Let $y = -x + 11$
    Then $x + y = 11 > 10$. $\square$
- Another successful proof!

Question of the Day

- Want to show that:

$$(\forall \epsilon > 0)(\exists \delta)(\forall (x, y) : \|(x, y) - (1, 1)\| < \delta)(|1 + x + y - 3| \leq \epsilon)$$

- The proof will look like:

    Proof Let $\epsilon > 0$
    Let $\delta =$
    Then (x, y) be such that $\|(x, y) - (1, 1)\| < \delta$.
    [Stuff happens]
    Then $1 + x + y \in [3 - \epsilon, 3 + \epsilon]$. $\square$

- To understand how to choose δ, we need to bound $|1 + x + y - 3|$ for $\|(x, y) - (1, 1)\| < \delta$. Now

$$\|(x, y) - (1, 1)\| < \delta \Rightarrow (x - 1)^2 + (y - 1)^2 < \delta,$$

  so $|x - 1| < \sqrt{\delta}$ and $|y - 1| < \sqrt{\delta}$.

- Equivalently, $x \in [1 - \sqrt{\delta}, 1 + \sqrt{\delta}]$, $y \in [1 - \sqrt{\delta}, 1 + \sqrt{\delta}]$.
- In inequality form:
$$1 - \sqrt{\delta} \leq x \leq 1 + \sqrt{\delta},\ 1 - \sqrt{\delta} \leq y \leq 1 + \sqrt{\delta}.$$
- Adding together and adding 1 gives
$$3 - 2\sqrt{\delta} \leq 1 + x + y \leq 3 + 2\sqrt{\delta}.$$
So $1 + x + y \in [3 - 2\sqrt{\delta}, 3 + 2\sqrt{\delta}]$.
- Want $1 + x + y \in [3 - \epsilon, 3 + \epsilon]$.
- Let $2\sqrt{\delta} = \epsilon$, so $\delta = (\epsilon/2)^2$, then result holds! Now to work in reverse and write out the steps of the proof one by one:

Proof Let $\epsilon > 0$
Let $\delta = (\epsilon/2)^2$
Then (x, y) be such that $\|(x, y) - (1, 1)\| < \delta$.
Then $(x - 1)^2 + (y - 1)^2 < \delta$.
So $|x - 1| \leq \sqrt{\delta} = \epsilon/2$ and $|y - 1| \leq \sqrt{\delta} = \epsilon/2$.
So $x \in [1 - \epsilon/2, 1 + \epsilon/2]$ and $y \in [1 - \epsilon/2, 1 + \epsilon/2]$.
Since $1 + x + y$ is an increasing function of x and y,
$1 + x + y \in [3 - \epsilon, 3 + \epsilon]$, and we are done! $\square$.

Limits for real-valued functions obey similar rules as for 1D limits.

Fact 6
Limits for functions from $\mathbb{R}^n$ to $\mathbb{R}$ are linear operators. So if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then for any $a, b \in \mathbb{R}$,
$$\lim_{x \to a} af(x) + bg(x) = aL + bM.$$

Problems

6.1: Prove the following:
$$(\exists x)(2x = 10)$$

6.2: Prove that $\lim_{(x,y) \to (0,0)} 1 - x + y = 0$.

7 Using partial derivatives of real-valued functions

Question of the Day Find the best linear approximation to

$$f(x, y) = x^2 y \text{ at } (x, y) = (3, 4).$$

Today

- Linear approximations of real-valued functions.
- Order of second derivatives doesn't matter
- Chain rule for curves

7.1 Linear approximations

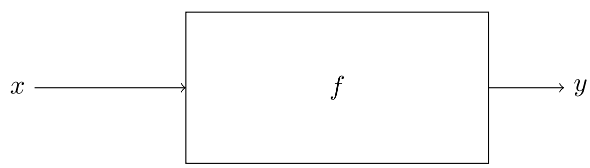
Recall differential conversions:

$dy = f' \; dx$	FTC
$ds = \|P'(x)\| \; dt$	arclength
$dy = \nabla f \cdot dx$	generalized FTC

For the last equation, $x = (x_1, x_2, \ldots, x_n)$. Note:

$$y + dy = f(x + dx),$$

which has picture:



so using the derivative:

$$y + \nabla f \cdot dx = f(x) + \nabla f \cdot dx$$

Setting $dx = h$ gives approximations:

$f_1(x + h) = f(x) + f'(x)h$	1 dimension
$f_1(x + h) = f(x) + \nabla f \cdot h$	n dimensions

Recall

- For $f : \mathbb{R} \to \mathbb{R}$, best linear approximation to f is

$$f_1(x + h) = f(x) + f'(x)h.$$

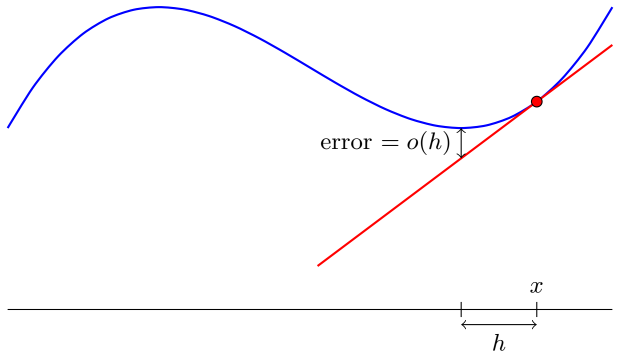
- When $h = 0$, gives $f_1(x) = f(x)$ and $f_1'(x) = f'(x)$.

Definition 37
Little o of $g(x)$ (written $o(g(x))$) is the set of functions

$$o(g(x)) = \left\{ f : \lim_{x \to 0} f(x)/g(x) = 0 \right\}.$$

- Ex: $x^2 \in o(x)$ since $\lim_{x \to 0} x^2/x = \lim_{x \to 0} x = 0$.
- Notation abuse: often write $x^2 = o(x)$ instead of $x^2 \in o(x)$.
- $f = o(g)$ means that f goes to 0 faster than g goes to 0.

Fact 7
For $f \in C^1$, $f(x+h) = f(x) + f'(x)h + o(h)$.



This is the best way of thinking of the derivative for functions $f : \mathbb{R}^n \to \mathbb{R}$:

Definition 38
Say that $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at x_0 if ∇f exists, and for $h \in \mathbb{R}^n$,

$$f(x+h) = f(x) + \nabla f(x) \cdot h + \|h\|\, g(h),$$

where $\lim_{h \to 0} g(h) = 0$.

Qotd

- Start with $f(x, y) = x^2 y$.
- Then $\nabla = (\partial/\partial x, \partial/\partial y)$, so
$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2xy, x^2).$$
- So
$$f_1(x+h) = f(x) + \nabla f \cdot h$$
which for $x = (3, 4)$ and $h = (h_x, h_y)$ makes
$$\begin{aligned} f((3+h_x, 4+h_y)) &= (3^2)(4) + (2 \cdot 3 \cdot 4, 3^2) \cdot (h_x, h_y) \\ &= \boxed{36 + 24h_x + 9h_y}. \end{aligned}$$
- I know this is a linear approximation because h_x and h_y are being raised to the 1st power.
- Some not linear functions of h_x and h_y:
$$h_x^2 + h_y, \quad \sin(h_x), \quad h_x h_y$$

7.2 Interchanging order of partial derivatives

Definition 39
For f that maps $(x_1, \ldots, x_n) \in \mathbb{R}^n$ to $y \in \mathbb{R}$, let D_i denote the partial derivative of f with respect to x_i.

- Ex: For $f(x, y, z)$, $D_2 f = \partial f / \partial y$.
- $D_2 f \cdot dy$ is about how much f changes when y is changed by dy
- $D_1 f \cdot dx$ is about how much f changes when x is changed by dx.

- So $D_2(D_1 f) \cdot dx\ dy$ is about how much f changes when x is changed by dx and then y changed by dy
- Also, $D_1(D_2 f) \cdot dy\ dx$ is about how much f changes when y is changed by dy and then x changed by dx
- So $f(x + dx, y + dy) = f(x, y) + D_2 D_1 f\ dx\ dy$ and
$f(x + dx, y + dy) = f(x, y) + D_1 D_2 f\ dx\ dy$.
- So $D_1 D_2 f = D_2 D_1 f$.

Fact 8
Let $f : U \to \mathbb{R}$ where $U \subset \mathbb{R}^2$ is open. If $D_1 f, D_2 f, D_1 D_2 f$, and $D_2 D_1 f$ exist and are continuous, then

$$D_1 D_2 f = D_2 D_1 f.$$

Example

$$f(x, y) = x^2 y + \sin(y),$$

$$D_1 f = \frac{\partial f}{dx} = 2xy \qquad D_2 f = \frac{\partial f}{\partial y} = x^2 + \cos(y)$$

$$D_2 D_1 f = \frac{\partial}{\partial y}(2xy) = 2x \qquad D_1 D_2 f = \frac{\partial}{\partial x}(x^2 + \cos(y)) = 2x.$$

Notation

$$\left(\frac{\partial}{\partial x}\right)^2 f = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}.$$

- This last in general does **not** equal $\left(\frac{\partial f}{\partial x}\right)^2$.
- Ex: $f(x, y) = x^2 y + \sin(y)$

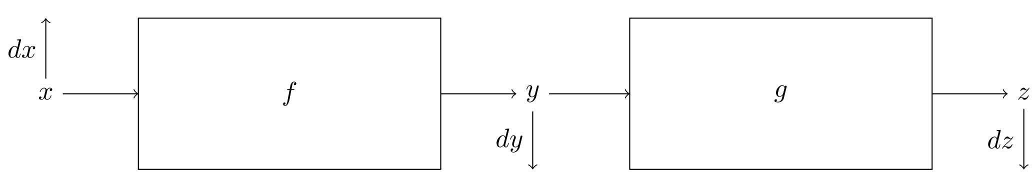
$$\frac{\partial f}{dx} = 2x \Rightarrow \left(\frac{\partial f}{\partial x}\right)^2 = 4x^2, \frac{\partial^2 f}{\partial x^2} = 2.$$

7.3 The Chain Rule

1D chain rule

$$y = f(x),\ z = g(y) = g(f(x)) = g \circ f(x)$$

In picture form:



Then:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

How much does z change when x changes? Change in x changes y, change in y changes z. Inputless way to say it:

$$[g \circ f]' = (g' \circ f) f'$$

- Ex: What is $d(\exp(x^2))/dx$?

- $g = \exp$, $f(x) = x^2$, $g' = \exp$, $f'(x) = 2x$, so

$$\frac{d(\exp(x^2))}{dx} = (g' \circ f)f' = \exp(x^2) \cdot 2x.$$

So how does it work for curves? Same idea, but ∇g for derivative of g, and C' for derivative of C.

Fact 9 (Chain Rule for curves)
Let $C(t) : \mathbb{R}^1 \to U$, $g : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ is open, $C, g \in C^1$. Then $(g \circ C)(t) = g(C(t)) \in C^1$, and

$$[g \circ C]' = ((\nabla g) \circ C) \cdot C'$$

Notation For $n = 3$:

$$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right), C'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

So

$$\nabla g \cdot C' = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt}.$$

- Ex: $C(t) = (e^t, t, t^2)$, $f(x, y, z) = x^2yz$. Find $\frac{d}{dt}f(C(t))$.
- Method 1: Chain Rule:

$$\frac{\partial f}{\partial x} = 2xyz, \quad \frac{\partial f}{\partial y} = x^2z, \quad \frac{\partial f}{\partial z} = x^2y.$$

$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 2t.$$

$$\begin{aligned}((\nabla f) \circ C)(t) \cdot C'(t) &= (2(e^t)(t)(t^2), (e^t)^2(t^2), (e^t)^2(t)) \cdot (e^t, 1, 2t) \\ &= 2e^{2t}t^3 + e^{2t}t^2 + e^{2t}(2t^2) \\ &= e^{2t}(2t^3 + 3t^2).\end{aligned}$$

- Note, don't actually need Chain Rule for curves to find derivative!
- Method 2: First find $f(C(t))$, then differentiate:

$$\begin{aligned}f(e^t, t, t^2) &= (e^t)^2(t)(t^2) = e^{2t}t^3 \\ [f(e^t, t, t^2)]' &= e^{2t}[t^3]' + [e^{2t}]'t^3 = 3t^2e^{2t} + 2t^3e^{2t}.\end{aligned}$$

Problems

7.1: (a) Find the best linear approximation for $f(x, y) = \sin(x + 2y)$ near $(\pi, 0)$.

(b) Use your approximation to estimate $\sin(x + 2y)$ at $(x, y) = (\pi + 0.1, 0.1)$.

8 Tangent Planes

Question of the Day Find the tangent plane to the surface $x^2 + y^2 + z^2 = 3$ at $(1, 1, 1)$.

Today

- Implicit and Explicit Function
- How to parameterize a plane
- Tangent plane as a linear approximation
- Directional derivatives

8.1 Explicit and Implicit Functions

A function can be thought of as a set of points:

- Example: $f(x) = x^2$ can be viewed as points (x, x^2) for $x \in \mathbb{R}$.
- Graph of the function is just the plot of this set of points.
- Say that $f(x) = x^2$ is an *explicit function.*

Another way to describe a set of points is by an equation that the points satisfy:

- Example: $x^2 + y^2 = 2$ is a circle of radius $\sqrt{2}$.

Definition 40
A set of points S is an **implicit function** if there is a function $f : \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$ such that

$$S = \{(x_1, x_2, \dots, x_n) : f(x_1, \dots, x_n) = c\}.$$

If $n = 2$ then it is an **implicit curve**, and if $n = 3$ it is an **implicit surface**.

- Example of explicit functions:

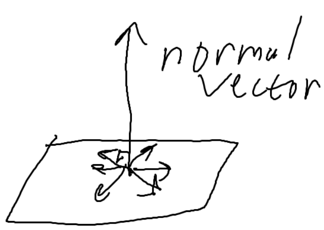
$$f(x) = x^2, \; g(x) = \sin(x), \; h(x) = x \exp(-x).$$

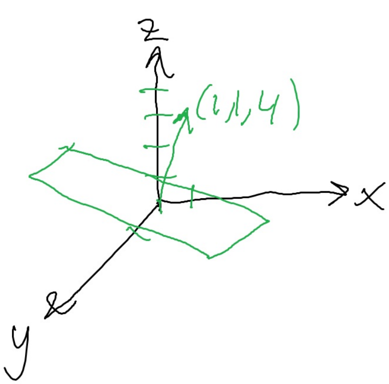
- Examples of implicit functions

$$x^2 + y^2 = 2, \; 3x + y^2 - z^2 = 1.$$

8.2 Parameterizing a plane

A plane can be described as the set of points that are perpendicular to a special vector, called the *normal vector*





- When $n = (0, 0, 1)$, plane is $x - y$ plane
- When $n = (1, 1, 4)$, plane is tilted slightly
- Recall: v and n are pependicular if $v \cdot n = 0$
- $n = (0, 0, 1)$, $(x, y, z) \cdot (0, 0, 1) = z$. So plane is $\boxed{z = 0}$
- $n = (1, 1, 4)$, $(x, y, z) \cdot (1, 1, 4) = x + y + 4z$. So plane is $\boxed{x + y + 4z = 0}$.
- Other direction: plane $3x-2y+z = 0$. Normal vector read from coefficients of equation: $n = (3, -2, 1)$.
- To make the plane go through points other than $(0, 0, 0)$, add a constant to right hand side.
- What is the equation of a plane with normal vector $(2, 1, -3)$ that passes through point $(1, 0, 1)$?
- Answer:
  - $2x + y - 3z = 0$ passes through $(0, 0, 0)$.
  - $2(1) + (0) - 3(1) = -1$
  - So $\boxed{2x + y - 3z = -1}$ passes through $(1, 0, 1)$ and has the correct normal vector.
- The plane defined by $2x + y - 3z = -1$ for some constant a, it an *implict* way of defining the plane. $z = (2/3)x + (1/3)y + (1/3)$ is an *explicit* way of defining the plane.

8.3 Tangent planes to implicit surfaces

- Want tangent plane to surface given by $f(x, y, z) = x^2 + y^2 + z^2 = 3$
- Note that the surface is defined implicitly.
- This is the surface of a sphere of radius $\sqrt{3}$.
- We know the plane is going to have the implicit form
$$g(x, y, z) = c_1x + c_2y + c_3z = c_4.$$
- To make derivatives match those of f, want $\nabla g = \nabla f$ at $(1, 1, 1)$.
- $\nabla f = (2x, 2y, 2z)$, so $\nabla f(1, 1, 1) = (2, 2, 2)$.
- Hence $g(x, y, z) = 2x + 2y + 2z$.
- How do we find c_4? Want $g(1, 1, 1) = 3$, so $2(1) + 2(1) + 2(1) = 6$.

Definition 41
The **tangent plane** to the implicit surface $f(v) = k$ at v_0 (where $f \in C^1$) is the implicit plane
$$\nabla f(v_0) \cdot (x, y, z) = \nabla f(v_0) \cdot v_0.$$

Example

- Example: Find the tangent plane to the implicit surface
$$xy + yz + zy = 6 \text{ at } (1, 6, 0).$$
- Steps:
  - Let $f(x, y, z) = xy + yz + xz$, $v_0 = (1, 6, 0)$
  - Find $\nabla f = (y + z, x + z, x + y)$
  - Find $\nabla f(v_0) = (6, 1, 7)$
  - Use formula: $(6, 1, 7) \cdot (x, y, z) = (6, 1, 7) \cdot (1, 6, 0)$ gives:
$$\boxed{6x + y + 7z = 12}$$

Same technique works for implicit curves

- Find the tangent line to the implicit curve $x^2y + y^3 = 10$ at $(1, 2)$.
- Steps
  - Let $f(x, y) = x^2y + y^3$, $v_0 = (1, 2)$
  - $\nabla f = (2xy, x^2 + 3y^2)$
  - $\nabla f(v_0) = (4, 13)$
  - Formula: $(4, 13) \cdot (x, y) = (4, 13) \cdot (1, 2)$:
  $$\boxed{4x + 13y = 30}.$$
  (This is the implicit equation of a line.)
- If the question was: find the explicit form of the tangent line for $x^2y + y^3 = 10$ at $(1, 2)$, the answer would be:
$$\boxed{y = -(4/13)x + (30/13).}$$

8.4 Directional derivatives

- Start with $f : \mathbb{R}^n \to \mathbb{R}$
- $D_i f$ is how fast f changes as variable x_i changes
- What if more than one variable changing, one variable could be changing faster than the other...
$$\begin{aligned} f(x + dx, y + dy, z + 2dz) &= f(x, y, z) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}(2dz) \\ &= f(x, y, z) + \nabla f \cdot (dx, dy, 2dz) \end{aligned}$$
- $\nabla f \cdot (dx, dy, 2dz)$ is the *directional differential*

Definition 42
Let $w \neq 0$. The **directional derivative** of $f \in C^1$ in the direction w at v is
$$D_w f(v) := \nabla f(v) \cdot w / \|w\|.$$

Fact 10
For $f \in C^1$, let $g(t) = f(v + tw)$. Then
$$D_w f(v) = g'(t).$$

Example:

- Let $f(x, y) = x^2 + y^3$ and let $w = (1, 2)$. Find the directional derivative of f in the direction w at $(-1, 3)$.
$$\begin{aligned} \nabla f(x, y) &= (2x, 3y^2) \\ \nabla f(-1, 3) &= (-2, 27) \\ \nabla f(-1, 3) \cdot \frac{w}{\|w\|} &= \frac{(-2, 27) \cdot (1, 2)}{\sqrt{1^2 + 2^2}} \\ &= \frac{-2 + 54}{\sqrt{5}} = \frac{52}{\sqrt{5}} \approx \boxed{23.25}. \end{aligned}$$

What maximizes the directional derivative?

- Directional derivative of the form $\nabla f \cdot w$. What direction for w maximizes the dot product?
- Recall: $\cos(\theta) = \frac{v \cdot w}{\|v\| \cdot \|w\|}$, so
$$v \cdot w = \cos(\theta) \|v\| \|w\| .$$
- So for $\|v\|$ and $\|w\|$ fixed, the maximum occurs when $\cos(\theta)$ is as large as possible, $\theta = 0$, $\cos(\theta) = 1$
- $\theta = 0$ means w and ∇f are pointing in the same direction.
- Idea: to increase f the fastest, move in the direction ∇f.
- For optimizing f, this gives the *steepest ascent* method.

Steepest Ascent step *Input:* $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, *Output:* $x \in \mathbb{R}^n$
1) Let $x \leftarrow x + \alpha \nabla f(x)$

Problems

8.1: Are the following sets of points written as explicit functions or as implicit functions?

(a) $y = 2x + 3$
(b) $x^2 + y^2 = 4$
(c) $z = x \exp(-xy)$
(d) $x \exp(-xy) - z = 0$

8.2: Find the tangent plane to $x^2 + y^2 + 2z^2 = 7$ at the point $(1, 2, 1)$

8.3: Find the tangent line to $x^3 - y^2 = -1$ at the point $(2, 3)$.

8.4: Find the directional derivative of $f(x, y) = (x^2, \exp(y))$ in the direction $(1, -1)$ from point $(2, 0)$.

9 Rotational Symmetry and the Laplace operator

Question of the Day Show that for $v = (x, y, z) \in \mathbb{R}^3$,

$$h(v) = \frac{-2k}{(x^2 + y^2 + z^2)^2}$$

is a rotationally symmetric function for any constant k.

Today

- Rotationally symmetric functions
- Harmonic functions

9.1 Rotational symmetry

Definition 43
The **distance function** is

$$r(x_1, x_2, \dots, x_n) = \sqrt{x_1^2 + \cdots + x_n^2} = \|(x_1, \dots, x_n)\|.$$

Definition 44
A real valued function is **rotationally symmetric** if

$$f(v) = g(r), \text{ where } r = \|v\|.$$

- So the output of rotationally symmetric functions only depends on the distance from the origin
- Ex: Gravity from the sun, electric field from an electron
- Ex: Light from a lamp
- Ex: probability density of independent normal random variables

Steepest ascent for rotationally symmetric functions

$$\begin{aligned}
\operatorname{grad} f = \nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
\frac{\partial f}{\partial x} = \frac{\partial g(r)}{\partial x} &= \frac{dg}{dr} \cdot \frac{\partial r}{\partial x} \\
&= g'(r) \cdot \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{1/2} \\
&= g'(r) \cdot 2x(1/2)(x^2 + y^2 + z^2)^{-1/2} = \frac{g'(r)}{r} x.
\end{aligned}$$

Can do same thing for y and z...

$$\frac{\partial f}{\partial y} = \frac{g'(r)}{r} y, \quad \frac{\partial f}{\partial z} = \frac{g'(r)}{r} z.$$

Putting it all together gives...

Fact 11
For a rotationally symmetric function $f(v) = g(r)$ (where $r = \|v\|$),

$$\nabla f(v) = \frac{g'(r)}{r} v.$$

- For rotationally symmetric functions, the steepest ascent direction is always directly away from the origin, or directly towards the origin.
- For gravity, a small object initally at rest, falls inward straight towards the larger object.
- To decrease the light received from a lamp the quickest, move directly away from the lamp.
- If $f(v) = g(r)$, then $\nabla f(v) = \frac{g'(r)}{r} v$. Here's the interesting part: its an iff statement. If $\nabla f(v) = (\frac{g'(r)}{/} r)$, then $f(v)$ is rotationally symmetric.

Fact 12 (Rotational symmetry)
A function f is rotationally symmetric if and only if $\nabla f(v) = h(r)v$ for some function h.

- Qotd: Show $h(v) = [-2k/(x^2 + y^2 + z^2)^2]$.
- Method 1: Write $h(v) = g(r)$.

$$r^2 = x^2 + y^2 + z^2 \Rightarrow h(v) = -2k/(r^2)^2 = -2k/r^4.$$

- Method 2: Find $\nabla h(v)$, show that it is proportional to v.

$$\begin{aligned}
\frac{\partial h(v)}{\partial x} &= -2k \cdot (-2)(2x)(x^2 + y^2 + z^2)^{-3} = -4kx/r^{3/2}, \\
\frac{\partial h(v)}{\partial y} &= -2k \cdot (-2)(2y)(x^2 + y^2 + z^2)^{-3} = -4ky/r^{3/2}, \\
\frac{\partial h(v)}{\partial z} &= -2k \cdot (-2)(2z)(x^2 + y^2 + z^2)^{-3} = -4kz/r^{3/2}, \\
\nabla h(v) &= -4k/r^{3/2} \, (x, y, z) \, .
\end{aligned}$$

9.2 Laplace operator

- Recall: $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$.
- So $\nabla f = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$
- Finally

$$\begin{aligned}
\nabla \cdot \nabla f &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.
\end{aligned}$$

Definition 45
The **Laplace operator** (written $\nabla \cdot \nabla$, ∇^2, or Δ), is given by

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2},$$

in n dimensions.

- The Laplace operator arises in steady state problems.
- For instance, suppose a drum has a skin stretched over a rim. If $f(x, y, t)$ is the height of the skin at location (x, y) at time t, then the physics of the problem gives the pde:

$$\frac{\partial^2}{\partial t^2} f = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = c^2 \nabla^2 f.$$

- Arises with fluid mixing to steady state as well.
- Shows up in image restoration models.

Definition 46
A function f is **harmonic** if
$$\nabla^2 f = 0$$

Example:

- Let $f(x, y) = 1/r^3$. Find Δf.
- Now $r = (x^2 + y^2 + z^2)^{1/2}$, so
$$\partial r/\partial x = (1/2)(x^2 + y^2 + z^2)^{-1/2}(2x) = x/r.$$
[Similarly,
$$\partial r/\partial y = y/r, \quad \partial r/\partial z = z/r.$$
- So
$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{1}{r^5}\frac{d(-3x)}{dx} + (-3x)\frac{\partial(r^{-5})}{\partial x} && \text{[Product Rule]} \\
&= \frac{1}{r^5}(-3) + (-3x)\frac{d(r^{-5})}{dr} \cdot \frac{\partial r}{\partial x} && \text{[Chain Rule]} \\
&= \frac{-3}{r^5} - 3x \cdot \frac{-5}{r^6}\frac{x}{r} \\
&= \frac{-3}{r^5} + \frac{15x^2}{r^7}.
\end{aligned}$$
- Repeating for y and z gives:
$$\begin{aligned}
\nabla^2 f &= \frac{15x^2}{r^7} - \frac{3}{r^5} + \frac{15y^2}{r^7} - \frac{3}{r^5} + \frac{15z^2}{r^7} - \frac{3}{r^5} \\
&= \frac{15(x^2 + y^2 + z^2)}{r^7} - \frac{9}{r^5} \\
&= \frac{6}{r^5}.
\end{aligned}$$

Problems

9.1: What is $\Delta(x^2 + y^2)$?

9.2: Be sure to justify your answers.

(a) Is $f(x, y, z) = (x^2 + y^2 + 2z^2)^{-1}$ rotationally symmetric?
(b) Is $f(x, y, z) = (x^2 + y^2 + z^2)^{-1}$ rotationally symmetric?
(c) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ can be written as $f(v) = 1/\|v\|^2$. Find the gradient of f.

10 Optimizing functions in 1 dimension

Question of the Day What is the maximum value of $f(x) = x \exp(-x)$ for $x \in [0, 3]$?

Today

- Boundedness theorem
- Extreme value theorem

10.1 Optimizing continuous functions over a compact set

- Closed bounded intervals have special properties

Definition 47
A set of real numbers A is **bounded** if there exists finite M such that $|a| \leq M$ for all $a \in A$. A set that is not bounded is called **unbounded**.

- Example: $[-3, 4]$, $(-3, 4]$ and $(-3, 4)$ are bounded
- Example: $(-\infty, 2)$ and $[0, \infty)$ are unbounded. (Basically look for the ∞ in one dimension.)

Definition 48
A set of real numbers A is **closed** if the complement of A, $A^C = \{a : a \notin A\}$, is open.

- Example: $(-\infty, 4]$ is closed, because the complement $(4, \infty)$ is open.
- Example: $(-3, 4)$ is not closed, because the complement $(-\infty, 3] \cup [4, \infty)$ is not open.
- Example: $\mathbb{R} = (-\infty, \infty)$ and $\emptyset$ are both open and closed. They are the only two sets that are both open and closed.

Definition 49
An interval that is both closed and bounded is called **compact**.

- When an interval is not compact, continuous functions can fly off to infinity (or negative infinity)
- Example: $f_1(x) = 1/x$ for $x \in (0, \infty)$ can be arbitrarily large
- Example: $f_2(x) = x$ for $x \in [1, \infty)$ can be arbitrarily large
- Both these examples have bounded range over compact intervals though.

Theorem 2 (Extreme Value Theorem)
Let f be continuous over the compact set A. Then

$$(\exists c, d \in A)(\forall x \in A)(f(c) \leq f(x) \leq f(d)).$$

That is, there exists c and d in A such that

$$\min_{x \in A} f(x) = f(c) \text{ and } \max_{x \in A} f(x) = f(d).$$

- In other words, there exists a point where the function f is maximized and where it is minimized.

10.2 Optimization with the extreme value theorem

- What input maximizes f over $[a, b]$?
- It could be at a or at b. Or it could be in (a, b).
- If f has a continuous derivative, then anywhere $f'(x) > 0$, f is strictly increasing, so it can't be a max or min when $f'(x) > 0$.
- Anywhere $f'(x) < 0$, f is strictly decreasing, can't be a max or min when $f'(x) < 0$.
- Max or min can only be at the endpoints of the interval or places where $f'(x) = 0$.

Definition 50
For $f : \mathbb{R} \to \mathbb{R}$, a **critical point** is any place where $f'(x) = 0$.

How to optimize continuous f over $[a, b]$

1: Find the critical points

2: Evaluate f at the critical points, and at the boundary of the interval at a and at b

3: The smallest function value must be the maximum of the function

4: The largest function value must be the minimum.

Qotd

- Find the critical points:

$$\begin{aligned}[x \exp(-x)]' &= [x]' \exp(-x) + x[\exp(-x)]' = \exp(-x) - x \exp(-x) \\ &= (1 - x) \exp(-x).\end{aligned}$$

- Recall, if $rs = 0$, either $r = 0$ or $s = 0$. So if $(1 - x) \exp(-x) = 0$, either $1 - x = 0$ or $\exp(-x) = 0$. Can't have $\exp(-x) = 0$, so only critical point is at $x = 1$.
- Make a table:

x	$f(x)$
0	0
1	$\exp(-1) \approx 0.3678\ldots$
3	$3\exp(-3) \approx 0.1493\ldots$

- Hence

$$\max_{x \in [0,3]} f(x) = 0.3678\ldots$$

- Not part of question, but also get:

$$\arg\max_{x \in [0,3]} f(x) = 1$$

and

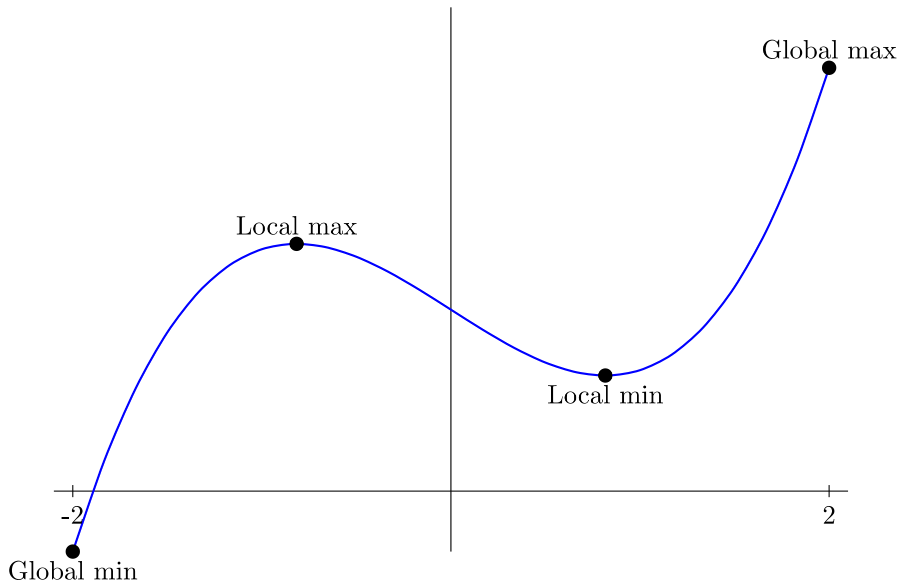
$$\min_{x \in [0,3]} f(x) = 0, \quad \arg\min_{x \in [0,3]} f(x) = 0$$

10.3 Global and local optima

Definition 51
Say that M is a **global maximum** for $f : A \to \mathbb{R}$ (write $M = \max_{x \in A} f(x)$) if there exists c such that $f(c) = M$, and for all $a \in A$, $f(a) \leq M$. Say that m is a **global minimum** for $f : A \to \mathbb{R}$ (write $m = \min_{x \in A} f(x)$) if

$$(\exists d)(\forall a \in A)(f(a) \geq f(d) = m)$$

- Global maxima are maxima where the function value is as large as any output.
- Local maxima are maxima where the function value is largest in a neighborhood of the input value.



Definition 52
Say (x, y) is a **local maximum** if $f(x) = y$, and

$$(\exists a, b : x \in (a, b)) \left(\max_{d \in (a,b)} f(d) = f(x) \right).$$

Definition 53
Say (x, y) is a **local minimum** if $f(x) = y$, and

$$(\exists a, b : x \in (a, b))(\min_{d \in (a,b)} f(d) = f(x)).$$

- Global optima are also local optima
- To find local optima, just find points with $f'(x) = 0$, and try to find a narrow enough $[a, b]$ so that $f'(x)$ is the global optima for that region.

Example

- Show that $(1, -3)$ is a local minimum of $x^3 - 3x - 1$.
- For $f(x) = x^3 - 3x - 1$, $f'(x) = 3x^2 - 3$, so the critical points are at $3x^2 - 3 = 0$, so x is either -1 or 1.
- So $[0, 2]$ only contains the critical point at $x = 1$.
- $f(0) = -1$, $f(1) = -3$, $f(2) = 1$, so $(1, -3)$ is a local minima.

Using second derivatives

- Note that at a local minima, $f'(x) = 0$, for $a < x$ $f'(x) < 0$, and for $a > x$, $f'(x) > 0$. So $f'(x)$ is increasing, which means $f''(x) \geq 0$.

Fact 13
If $f'(x) = 0$ and $f''(x) > 0$, then $(x, f(x))$ is a local minimum of the function f. If $f'(x) = 0$ and $f''(x) < 0$, then $(x, f(x))$ is a local maximum of the function f.

Example

- Show that $(1, -3)$ is a local minimum of $f(x) = x^3 - 3x - 1$
- $f'(x) = 3x^2 - 3$, $f''(x) = 6x$.
- $f(1) = 1 - 3 - 1 = -3$, $f'(1) = 3(-1)^2 - 3 = 0$, $f''(1) = 6(1) = 6 > 0$.
- So $(1, -3)$ is a local minimum

Problems

10.1: Which of the following sets are bounded? (You do not have to prove your answer.)

(a) $[4, \infty)$
(b) $(-\infty, \infty)$
(c) $[0, 3)$

10.2: Suppose $f(x) = x^3 - x$.

(a) Find $\max_{x \in [-1,2]} f(x)$.
(b) Find $\arg\max_{x \in [-1,2]} f(x)$.
(c) Find $\min_{x \in [-1,2]} f(x)$.
(d) Find $\arg\min_{x \in [-1,2]} f(x)$.

11 Optimizing functions in n dimensions

Question of the Day Find
$$\max_{A \in \{(x,y):x^2+y^2 \leq 1\}} e^{-(x^2+y^2)}.$$

Today

- Optimization
- Global and local optima
- Critical points
- Closed, bounded sets

11.1 Optimization over closed, bounded sets

- Intuition: v is on the boundary of the set S if any small circle around v contains points both in the set and out of the set.

Definition 54
Say that v is on the **boundary** of S if
$$(\forall a > 0)(\exists w : \|w - v\| \leq a, w \in S)(\exists w' : \|w' - v\| \leq a, w \notin S).$$

- Note Open sets (by definition) cannot contain their boundary points.
- Ex: boundary points of open interval $(3, 4)$ is 3 and 4. [This is also the boundary of $[3, 4]$ and $(3, 4]$.]
- Ex: boundary points of $\{(x, y) : x^2 + y^2 \leq 1\}$ is the points $\{(x, y) : x^2 + y^2 = 1\}$.
- Note: boundary of $\{(x, y) : x^2 + y^2 < 1\}$ same as boundary of $\{(x, y) : x^2 + y^2 \leq 1\}$.
- Ex: $(-\infty, \infty) = \mathbb{R}$ has no boundary points!

Fact 14
A set is closed iff it contains all of its boundary points.

- Ex: $[3, 4]$ is closed, while $(3, 4]$ is not (in both cases $\{3, 4\}$ are the boundary points.
- Ex: $\{(x, y) : x^2 + y^2 \leq 1\}$ is closed, but $\{(x, y) : x^2 + y^2 < 1\}$ is not.

Definition 55
A set $S \subseteq \mathbb{R}^n$ is **bounded** if
$$(\exists M)(\forall s \in S)(\|s\| \leq M).$$
A set that is not bounded is **unbounded**.

- Note: the words bounded and boundary sound alike, but have very different meanings. Do not confuse the two!
- Ex: $[3, 4]$ is bounded $(M = 4, M = 5)$, $\{(x, y) : x^2 + y^2 \leq 1\}$ is bounded $(M = 1)$.
- Ex: $(-\infty, 3)$ is not bounded (unbounded).

Definition 56
A set that is closed and bounded is called **compact**.

Extreme value theorem from before still holds with more general definition of compact in $\mathbb{R}^n$.

- Suppose A is compact and $f : A \to \mathbb{R}$ is continuous.
- Then there exists c and d in A such that

$$\min_{x \in A} f(x) = f(c) \text{ and } \max_{x \in A} f(x) = f(d)$$

- So to optimize f over a compact set A...
- Use same idea as in 1 dimension

  1: Optimize over the boundary of A

  2: Optimize over all points in A not in the boundary

  3: Take the best value that results.

11.2 Finding critical points

Definition 57
A point in a set A that is not on the boundary of A is called an **interior point**.

Fact 15
The interior of A is always an open set.

Definition 58
Say that v is a **critical point** or **stationary point** of f if $\nabla f(v) = 0$.

Why are critical points important

- Turns out, local optima $(v, f(v))$ always have v a critical point.

Reasoning behind the fact

- Remember our first order linear approximation of the function:

$$f(v + h) \approx f(v) + \nabla f(v) \cdot h$$

  If $\nabla f(v) \neq (0, 0, \ldots, 0)$, set $h = \alpha \nabla f(v)$ (where $\alpha > 0$), so

$$\nabla f(v) \cdot h = \alpha \nabla f(v) \cdot \nabla f(v) = \alpha \left\| \nabla f(v) \right\|^2 > 0.$$

  Then $f(v + h) \approx f(v) + \text{something positive}$, so $f(v)$ cannot be the maximum value.

- Similarly, set $\alpha < 0$ to show that $f(v)$ not the minimum value.
- Ex: $f(x, y) = \exp(-(x^2 + y^2))$:

$$\nabla f = \left(-2xe^{-(x^2+y^2)}, -2ye^{-(x^2+y^2)} \right).$$

- $e^{-w} \neq 0$, so only critical point has $(-2x, -2y) = (0, 0)$, so $x = 0$, $y = 0$. Unique critical point at $(0, 0)$.

Definition 59
Point v is a **global maximum** for $f : A \to \mathbb{R}$ if

$$(\forall w)(f(w) \leq f(v)).$$

Point v is a **global minimum** if $(\forall w)(f(w) \geq f(v))$.

Definition 60
Point v in an open set U is a **local maximum** if

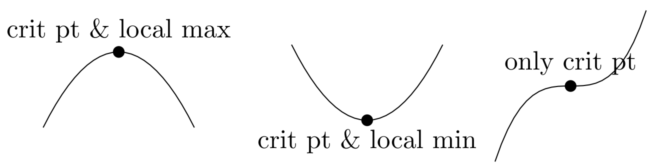
$$(\exists a)(\forall w : \|v - w\| < a)(f(w) \leq f(v)).$$

(It is a **local minimum** if last clause is $f(w) \geq f(v)$.)

- A local max is where all nearby points have either the same or smaller function value.

Theorem 3 (Critical point theorem for n dimensions)
If v is a local maximum or minimum for $f : A \to B$ for $f \in C^1$, and A open, then v is a critical point.

- Critical points work similar to how they do in one dimension.
- All local min/max are critical points
- Not all critical points are local min/max.



Pf: Suppose v is a local maximum.
Let $w \neq 0$.
Since U is open, let t_1 be a small enough value such that $(\forall t \leq t_1)(v + tw \in U)$.
Since v is a local maximum, let t_2 be a small enough
value such that $(\forall t \leq t_2)(f(v + t_2 w) \leq f(v)$.
Let $g(t) = f(v + tw)$
Then $g(t)$ has a local maximum at $t = 0$.
Calc I result: that means $g'(t) = 0$.
The chain rule for curves:
$g'(t) = \dfrac{dg}{dt} = \dfrac{df(v + tw)}{dt} = \nabla f \cdot w.$
So $\nabla f \cdot w$ for all $w \neq 0$.
The only way that can happen is if $\nabla f = 0$. $\square$

Qotd

- For qotd, only critical point in the interior is $(0, 0)$, $f(0, 0) = 1$.
- On the boundary, $x^2 + y^2 = 1$, so $f(x, y) = \exp(-1)$. Since $\exp(-1) < 1$,

$$\max_{(x,y):x^2+y^2\leq 1} x^2 + y^2 = 1.$$

Problems

11.1: Which of the following sets are bounded? (You do not have to prove your answer.)

(a) $\{(x, y) : x^2 + 2y^2 \leq 4\}$

(b) $\{(x, y) : x^2 \geq y\}$

(c) $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$

11.2: Suppose $f(x, y) = \exp(-x^2 - 2y^2)$. Let $A = \{(x, y) : x^2 + y^2 \leq 4\}$.

(a) Find $\max_A f(x, y)$.

(b) Find $\arg\max_A f(x, y)$.

(c) Find $\min_A f(x, y)$.

(d) Find $\arg\min_A f(x, y)$.

12 Optimization over noncompact regions

Question of the Day What is

$$\max_{(x,y)\in\mathbb{R}^2} \exp(-(x^2+y^2))?$$

Today

- Optimizing over noncompact regions

Optima do not always exist

- Extreme value theorem says that continuous functions guaranteed to have optima over compact set.
- When set not compact, could go either way.
- Ex: $\max_{x\in[0,\infty)} x$ does not exist.
- Ex: $\max_{x\in[0,\infty)} \exp(-x) = \exp(0) = 1$ since $\exp(-x)$ is a decreasing function.
- Can these problem be approached systematically?
- Find $\max_{x\geq 0} x\exp(-x)$.

12.1 Optimization in 1-D

- Assume that an interval is closed but unbounded
- Then best way to optimize is to show that the function is decreasing or increasing, and optimize accordingly.

Definition 61
A function $f : \mathbb{R} \to \mathbb{R}$ is **increasing**

$$(\forall a < b)(f(a) \leq f(b)).$$

If $(\forall a < b)(f(a) < f(b))$, then the function is **strictly increasing** A function is **decreasing** if $(\forall a < b)(f(a) \geq f(b))$ and **strictly decreasing** if $(\forall a < b)(f(a) > f(b))$.

Fact 16
If $f'(x) \geq 0$ for all $x \in [a,b]$, then $f(x)$ is increasing over $[a,b]$. (For $f'(x) > 0$, f is strictly increasing.) Similarly, if $f'(x) \leq 0$ for all $x \in [a,b]$, then $f(x)$ is decreasing over $[a,b]$. (For $f'(x) < 0$, f is strictly decreasing.)

- With increasing functions f, the minimum is when x is as small as possible.

Fact 17
For increasing or decreasing functions f:

1: If $f(x)$ is increasing over $[a,\infty)$ then $\min_{x\in[a,\infty)} f(x) = f(a)$.

2: If $f(x)$ is increasing over $(-\infty,b]$ then $\max_{x\in(-\infty,b]} f(x) = f(b)$.

3: If $f(x)$ is decreasing over $[a,\infty)$ then $\max_{x\in[a,\infty)} f(x) = f(a)$.

4: If $f(x)$ is decreasing over $(-\infty,b]$ then $\min_{x\in(-\infty,b]} f(x) = f(b)$.

- If $f'(x)$ is not $\geq$ or ≤ 0 over an unbounded region, break region apart.

Fact 18
Suppose $\text{opt}_{x \in A} f(x) = M_A$ and $\text{opt}_{x \in B} f(x) = M_B$. Then

$$\text{opt}_{x \in A \cup B} f(x) = \text{opt}\{M_A, M_B\}.$$

Example:

- Find $\max_{x \in [0,\infty)} x \exp(-x)$.
- For $f(x) = x \exp(-x)$, $f'(x) = \exp(-x) - x \exp(-x) = \exp(-x)(1 - x)$.
- Since $\exp(-x) > 0$ for all x, $f(x) \geq 0$ for $x \leq 1$, $f(x) \leq 0$ for $x \geq 1$. So

$$\max_{x \in [0,1]} f(x) = f(1) = \exp(-1), \quad \max_{x \in [1,\infty)} f(x) = f(1) = \exp(-1),$$

which means

$$\max_{x \in [0,\infty)} f(x) = \max_{x \in [0,1] \cup [1,\infty)} f(x) = \max\{\exp(-1), \exp(-1)\}.$$

12.2 Using covers to optimize over unbounded regions in n dimensions

- Try to cover noncompact region by sets that are compact.

Definition 62
Let $\{A(a)\}$ be a collection of sets such that $A \subseteq \cup_a A(a)$. Then say that $\{A(a)\}$ is a **cover** of A.

Fact 19
Let $\{A(a)\}$ be a collection of compact sets that cover $\mathbb{R}^n$. Further, suppose

$$(\exists v \in \mathbb{R}^n)(\forall a)(\max_{x \in A(a)} f(x) \leq f(v)).$$

Then

$$\max_{w \in \mathbb{R}^n} f(w) = f(v).$$

What is global max over $\mathbb{R}^2$?

- Let $A(a) = \{(x, y) : x^2 + y^2 \leq a\}$.
- Then $(0, 0)$ only critical point inside $A(a)$. $f(0, 0) = \exp(-0) = 1$
- On boundary, $x^2 + y^2 = a$ so $f(x, y) = \exp(-a) \leq 1$.
- So $\max_{x \in A(a)} f(x) = 1$ for all a.
- Hence $\max_{x \in \mathbb{R}^n} f(x) = 1$.

Problems

12.1: (a) What is $\max_{x \in [0,\infty)} x^2 \exp(-2x)$?
(b) What is $\max_{x \in (-\infty,\infty)} 3 - x^2$?
(c) What is $\min_{x \in (-\infty,\infty)} |x|$?

13 Constrained Maximization

Question of the Day Maximize $f(x, y) = x + y$ subject to $x^2 + 2y^2 \leq 1$.

Today

- Constraints given by implicit functions in C^1

Constraints

- The economic problem: all resources are finite
- Ex: time, money, oil, unobtainium
- The mathematics problem: how to optimize the objective given constraints

Recall

- Recall *explicit functions* tell you how to calculate outputs from inputs. For $f : \mathbb{R}^n \to \mathbb{R}^m$:

$$(y_1, \ldots, y_m) = f(x_1, \ldots, x_n)$$

- *implicit functions* are a set of points $(x_1, \ldots, x_n)$ such that $f(x_1, \ldots, x_n) = c$ for a fixed constant c.
- The boundary of sets is often given by an implicit function

$$A = \{(x, y) : x^2 + y^2 \geq 2\}, \quad \partial A = \{(x, y) : x^2 + y^2 = 2\}$$

  (Here ∂A is another notation for the boundary of A.)

Examples

- Explicit: $y = x^2$, Implicit: $y - x^2 = 0$. In fact, every explicit function $y = f(x)$ has an implicit function description $y - f(x) = 0$.
- Implicit: $x^2 + y^2 = 1$. In this case, there is no explicit function description of this set of points.

So far

- For compact set: Optimum value either at boundary or interior.
- Qotd: Either $\{(x, y) : x^2 + 2y^2 < 1\}$ or $\{(x, y) : x^2 + 2y^2 = 1\}$.
- For $f \in C^1$, interior values where optima occur are critical points

$$\nabla f = (1, 1),$$

  so no critical points in interior!
- So that leaves the boundary to consider:

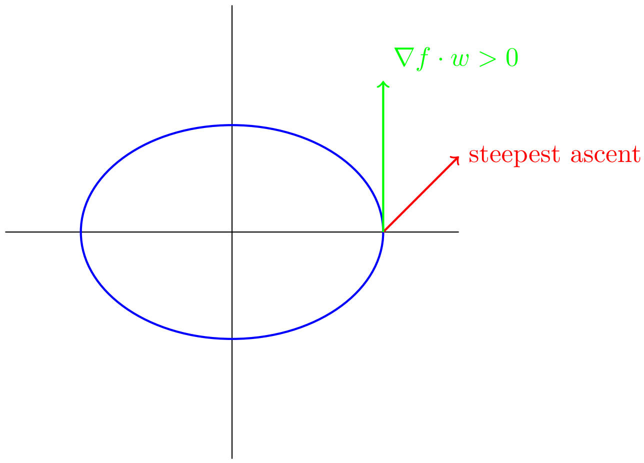
$$\max_{x^2+2y^2=1} x + y = ?$$

- Note that the boundary points can be written as an implicit function.

13.1 Lagrange Multipliers

Recall that ∇f is direction of steepest ascent, that is, the direction to travel in such that f is increasing most rapidly.

- But we cannot move freely in the contrained case, we are limited to moving on boundary. For the question of the day, suppose we start at $(1,0)$ Then $\nabla(x,y) = (1,1)$. Want to move in this direction, but that would take us off of the ellipse.



- Recall
$$f(v+tw) \approx f(v) + t(\nabla f \cdot w)/\|w\| = f(v) + tD_w(f(v)).$$
As long as $\nabla f \cdot w > 0$, still can improve answer.

- Recall tangent line to curve $g(x,y) = x^2 + y^2 = 1$ at (x_0, y_0) is
$$\nabla g \cdot (x,y) = \nabla g \cdot (x_0, y_0).$$
∇g is perpendicular/normal to the tangent line.

- If both ∇f and ∇g are perpendicular to tangent line, then they are pointing in the same direction!
$$\nabla f = \lambda \nabla g$$
for some constant λ.

Definition 63
If $\nabla f = \lambda \nabla g$, call λ a **Lagrange multiplier**.

Definition 64
Say that v_c is a **critical point** for $\text{opt}_{g(v)=0} f(v)$ if $\nabla f(v_c) = \lambda \nabla g(v_c)$ for a constant $\lambda \in \mathbb{R}$.

- Critical point for $\max_{x^2+2y^2=1} x + y$ satisfy
$$(2x, 4y) = \lambda(1,1).$$
Gives three equations (two from $(2x,4y) = \lambda(1,1)$ and one from implicit function)
$$2x = \lambda, \ 4y = \lambda, \ x^2 + 2y^2 = 1.$$

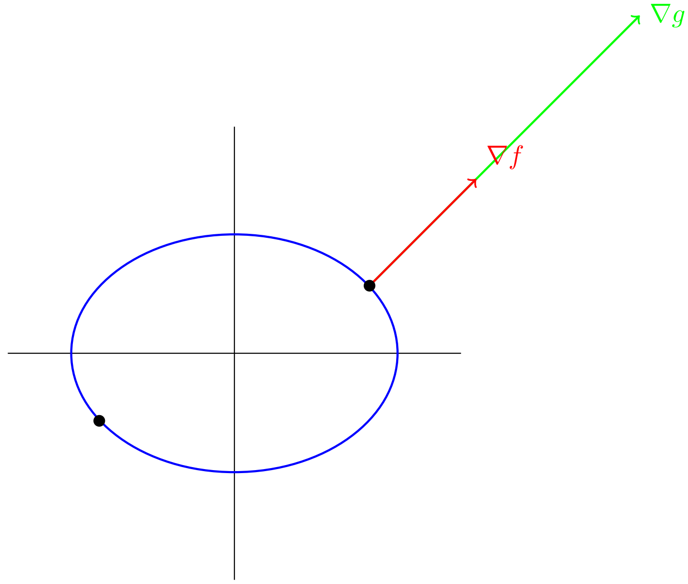
- Don't care about λ, so try to eliminate in equations
$$2x = \lambda = 4y \Rightarrow 2x = 4y \Rightarrow x = 2y.$$

- Implicit equation then gives

$$(2y)^2 + 2y^2 = 1 \Rightarrow 5y^2 = 1 \Rightarrow y = \pm 1/\sqrt{5}.$$

Two critical points: $(2/\sqrt{5}, 1/\sqrt{5}), (-2/\sqrt{5}, -1/\sqrt{5})$.

$$f(2/\sqrt{5}, 1/\sqrt{5}) \approx 1.341, \quad f(-2/\sqrt{5}, -1/\sqrt{5}) \approx -1.341.$$



Theorem 4 (Optimization with a constraint)
If v is a local maximum or minimum for $\text{opt}\{f(v)|g(v) = 0\}$, where $f, g \in C^1$, then v is a critical point where $\nabla f = \lambda \nabla g$ for some constant $\lambda \neq 0$.

- So if constraint g is continuous, solve $\text{opt}_{v:g(v)=0} f(v)$ as follows:
  - Find ∇g and ∇f
  - Solve $n + 1$ equations:

$$\underbrace{\nabla f = \lambda \nabla g}_{n \text{ equations}}, \; \underbrace{g(v) = 0}_{1 \text{ equation}}.$$

- Note that you can always write the constraint in the form $g(v) = 0$ because if $g(v) = c$ just subtract c from both sides of the equation.
- For example: constraint $x^2 + 2y^2 = 1$, $g(x, y) = x^2 + 2y^2 - 1$.

Problems

13.1: Solve the following optimzation problems.

(a) Find $\max\{x + y^2 | 2x^2 + y^2 \leq 2\}$.

(b) Find $\min\{x + y^2 | 2x^2 + y^2 \leq 2\}$.

(c) Find $\arg\max\{x + y^2 | 2x^2 + y^2 \leq 2\}$.

14 Constrained Maximization without continuous derivatives

Question of the Day Find

$$\max_{0 \le x \le 1, 0 \le y \le 1} x^2 y.$$

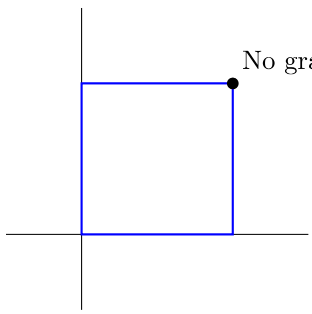
Today

- Constraints where the boundary does not have a continuous derivative

Thinking about the question of the day...

$$\max_{(x,y) \in [0,1] \times [0,1]} x^2 y.$$

The region over which we are maximizing is a square



- As usual, write $A = \text{int}(A) \cup \partial A$.

$$\text{int}(A) = (0,1) \times (0,1)$$
$$\partial A = \underbrace{(\{0\} \times [0,1])}_{\text{left}} \cup \underbrace{([0,1] \times \{0\})}_{\text{bottom}} \cup \underbrace{(\{1\} \times [0,1])}_{\text{right}} \cup \underbrace{([0,1] \times \{1\})}_{\text{right}}.$$

  Note $\{0\} \times [0,1] = \{(x,y) : x = 0, y \in [0,1]\}$, so left side of square.

- Remember, if you can write $A = A_1 \cup A_2 \cup \cdots \cup A_n$, then

$$\max_{v \in A} f(v) = \max\{\max_{v \in A_1} f(v), \ldots, \max_{v \in A_n} f(v)\}.$$

- Here

$$A_1 = \text{int}(A), \; A_2 = \text{left}, \ldots, A_5 = \text{right}$$

- Do each max problem separately. To find $\max_{v \in \text{int}(A)} f(x)$, look at critical points:

$$\nabla f(x,y) = (2xy, x^2) = (0,0).$$

  Since $x^2 = 0 \Rightarrow x = 0$, there are no critical points in $\text{int}(A)$!

- When $x = 0$, $f(0,y) = 0$. When $x = 1$, $f(1,y) = y$, so

$$\max_{y \in [0,1]} f(1,y) = \max_{y \in [0,1]} y = 1.$$

- Similarly, when $y = 0$, $f(x,0) = 0$. When $y = 1$, $f(x,1) = x^2$, so

$$\max_{x \in [0,1]} f(x,1) = \max_{x \in [0,1]} x^2 = 1.$$

- Put all the pieces together to get:

$$\max_{(x,y) \in A} f(x) = \max\{0,0,1,1\} = \boxed{1}$$

Example On a budget of \$90 million, a company buys model A1000 at \$3 mil/unit, model B1000 at \$5 mil/unit. If the company buys x units of the A1000's and y units of the B1000's, then utility (reward) is xy. Find x and y to maximize the utility.

- Solution: first pretend can buy fractional numbers of units.
- To write the constraint, consider everything in terms of millions of dollars.
- Three constraints: $3x + 5y \leq 90$, $x \geq 0$, $y \geq 0$, since we can't buy negative amounts of the units.
- Want to know
$$\arg\max_{3x+5y\leq 90, x\geq 0, y\geq 0} xy.$$
- $\nabla f = (y, x) = (0, 0) \Rightarrow (x, y) = (0, 0)$, so no critical points in interior.
- $x = 0$ or $y = 0$ gives $f = 0$, so don't have to worry about this part of boundary.
- Remaining part $3x + 5y = 90$. $g(x, y) = 3x + 5y - 90 = 0$. $\nabla g = (3, 5)$. Lagrange multipliers gives 3 equations:
$$\begin{aligned} y &= 3\lambda \\ x &= 5\lambda \\ 3x + 5y &= 90 \end{aligned}$$
- Solving $\lambda = y/3 = x/5 \Rightarrow y = (3/5)x$, $3x + 3x = 90 \Rightarrow x = 15 \Rightarrow y = 9$.
- Since x and y are both integers, must be best choice.

Problems

14.1: Graph $\{(x, y) : x^3 - y^2 = 0, x \in [-1, 1]\}$

14.2:
(a) Find $\max\{x^2 + y | x^3 - y^2 = 0, x \in [-1, 1]\}$
(b) Find $\arg\max\{x^2 + y | x^3 - y^2 = 0, x \in [-1, 1]\}$
(c) Find $\min\{x^2 + y | x^3 - y^2 = 0, x \in [-1, 1]\}$
(d) Find $\arg\min\{x^2 + y | x^3 - y^2 = 0, x \in [-1, 1]\}$

14.3:
(a) Find $\max_{x^2-1\leq y\leq 1-x^2} x^2 - 3y^2$
(b) Find $\min_{x^2-1\leq y\leq 1-x^2} x^2 - 3y^2$

15 Matrices and linear transformations

Question of the Day Suppose f is the map $(x,y) \mapsto (3x-y, 2y, x+y)$ and g is the map $(x,y,z) \mapsto (x-y, x+z)$. What is $[g \circ f](x,y) = g(f(x,y))$?

Today

- Linear tranformations
- Matrices
- Matrix multiplication

Recall

- $f : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear operator* or *linear transformation* if

$$(\forall v, w \in \mathbb{R}^n)(\forall \alpha, \beta \in \mathbb{R})(f(\alpha v + \beta w) = \alpha f(v) + \beta f(w))$$

- Claim: $f(x,y) = (3x-y, 2y, x+y)$ is a linear operator.

  Proof. Let $v = (x_1, y_1), w = (x_2, y_2) \in \mathbb{R}^2$, let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} f(\alpha v + \beta w) &= f(\alpha(x_1,y_1) + \beta(x_2,y_2)) \\ &= f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \\ &= (3(\alpha x_1 + \beta x_2) - (\alpha y_1 + \beta y_2), \alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2) \\ &= \alpha(3x_1 - y_1, x_1 + y_1) + \beta(3x_2 - y_2, x_2 + y_2) \\ &= \alpha f(x_1, y_1) + \beta f(x_2, y_2). \end{aligned}$$

□

General linear operators

- Why does this work? $3x-y$ and $x+y$ work because x and y are being raised to the first power. Can check: $3xy$ would *not* give a linear operator.
- Names of variables don't matter: $f(a,b) = (3a-b, a+b)$ is same transformation.
- Therefore all that matters is the coefficients in front of the variables $(3,-1)$ for $3x-y$, and $(1,1)$ for $x+y$.
- When listed as rows of a table, these form a *matrix*.

Definition 65
A **matrix** is a table of entries. An n by m (also written $n \times m$) size matrix has n rows and m columns. The i,jth entry of the matrix is the entry in the ith row and jth column.

Examples These are all matrices:

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} x & y \\ y & z \\ z & x \end{pmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 4 & 2 & 0 \end{bmatrix}$$

Using matrices to represent linear transformations

- Note that $3x - y = (3, -1) \cdot (x, y)$.

Definition 66
Let $v \in \mathbb{R}^n$ be a vector. Say that v is a **column vector** if it is written as a matrix with n rows and 1 column, and a **row vector** if it is written with 1 row and n columns.

Examples

- These are column vectors:

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 4 \end{pmatrix}$$

- These are row vectors

$$\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} x & y \end{pmatrix}, \begin{pmatrix} 4 \end{pmatrix}$$

15.1 Multiplying matrices by vectors

- Linear transformations represented by a new form of "multiplication"
- Recall $f(x, y, z) = 3x + 2y - z$ can be represented by matrix

$$A = \begin{pmatrix} 3 & 2 & -1 \end{pmatrix}$$

- For $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, want $Av = f(v)$. So

$$\begin{pmatrix} 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} := 3x + 2y - z.$$

Definition 67

A row matrix $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$ **times** a column matrix $v = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is defined to be the same as the dot product:

$$Av = a_1 b_1 + \cdots + a_n b_n$$

- Only defined for $1 \times n$ times $n \times 1$ matrix so far.

Fact 20
Any $f : \mathbb{R}^n \to \mathbb{R}$ that is a linear operator is of the form $f(v) = Av$ for some $1 \times n$ matrix A.

- What if $f : \mathbb{R}^n \to \mathbb{R}^m$? Ex:

$$f(x, y, z) = \begin{pmatrix} 3x + 2y - z \\ x - y + z \end{pmatrix}$$

- Each row gives one row of the matrix A:

$$f(v) = Av = \begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Definition 68
Let A be an $m \times n$ matrix and v a column vector with n entries. Then if A has rows $r_1, r_2, \ldots, r_m$, then

$$Av = \begin{pmatrix} r_1 \cdot v \\ r_2 \cdot v \\ \vdots \\ r_m \cdot v \end{pmatrix}.$$

Fact 21
Any $f : \mathbb{R}^n \to \mathbb{R}^m$ that is a linear operator is of the form $f(v) = Av$ for some $m \times n$ matrix A.

We will not give the proof of this important fact here, it is typically given in a course in linear algebra.

15.2 Multiplying matrices by matrices

- Now suppose $f : \mathbb{R}^m \to \mathbb{R}^n$, and $g : \mathbb{R}^n \to \mathbb{R}^s$. Then $g \circ f : \mathbb{R}^m \to \mathbb{R}^s$.
- Recall $[g \circ f](v) = g(f(v))$.

Fact 22
Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^s$ are linear operators. Then so is $g \circ f$.

Proof. Let $v, w \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} [g \circ f](\alpha v + \beta w) &= g(f(\alpha v + \beta w)) \\ &= g(\alpha f(v) + \beta f(w)) \\ &= \alpha g(f(v)) + \beta g(f(w)) \\ &= \alpha [g \circ f](v) + \beta [g \circ f](w). \end{aligned}$$

□

- That means if $f(v) = Av$ where A is $m \times n$, and $g(w) = Bw$ where B is $s \times m$, then $g(f(v)) = Cv$ where C is an $s \times n$ matrix.

Definition 69
If $f(v) = Av$ where A is $m \times n$ and $g(w) = Bw$ where B is $s \times m$, then $C = BA$ is defined to be the matrix such that $g(f(v)) = Cv$.

Fact 23
For B an $s \times m$ matrix and A an $m \times n$ matrix, the matrix $C = BA$ is a matrix whose i, jth entry is $r_i \cdot c_j$, where r_i is the ith row of matrix B, and c_j is the jth column of matrix A.

Qotd: First write f and g using matrices, then multiply the matrices together.

$$\begin{aligned} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} &= \begin{pmatrix} (1,-1,0) \cdot (3,0,1) & (1,-1,0) \cdot (-1,2,1) \\ (1,0,1) \cdot (3,0,1) & (1,0,1) \cdot (-1,2,1) \end{pmatrix} \\ &= \begin{pmatrix} 3 & -3 \\ 4 & 0 \end{pmatrix} \end{aligned}$$

So that means

$$g(f(x, y)) = (3x - 3y, 4x).$$

[Can check that is true directly as well.]

Notes

- In general for two functions g and f the order of composition matters. That is, $g(f(v)) \neq f(g(v))$. In the same way, for two matrices $AB \neq BA$ in general. (If A or B is not square, then one direction won't even be defined!)
- The dimension of the output of the first transforation has to equal the dimension of the input of the second transformation. For matrices, that means that in order to multiply them, the inner dimensions must match:

$$\begin{aligned}(3 \times 4)(4 \times 7) &= (3 \times 7)\\(x \times y)(y \times z) &= (x \times z)\\(5 \times 4)(5 \times 4) &= \text{undefined}\end{aligned}$$

Problems

15.1: Multiply the following row vectors times column vectors:

(a) $\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(b) $\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

15.2: Calculate the following products of matrices.

(a) $\begin{pmatrix} 2 & 4 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & -2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

15.3: What is $\begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}^2$?

16 Higher Derivatives

Question of the Day Find the best quadratic approximation to the function $f(x,y) = x^3y$ at $(1,1)$.

Today

- 2nd derivative of a real-valued function: Hessian

16.1 Second order approximations

Taylor series

- First degree Taylor polynomial (linear approx):

$$f_1(x_0 + h) = f(x_0) + hf'(x_0)$$

- Second degree Taylor polynomial (quadratic approx)

$$f_2(x_0 + h) = f(x_0) + hf'(x_0) + (1/2!)h^2 f''(x_0)$$

Want to extend this to higher dimensions. We need to add up all the changes to the function value coming from the first dimension, and the second dimension.

For $f : \mathbb{R}^n \to \mathbb{R}^m$:

$$\begin{aligned} f_2(v_0 + h) &= f(v_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h(i) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} h(i)h(j) \\ &= f(v_0) + \sum_{i=1}^{n} D_i f(v_0)h(i) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} D_i D_j f(v_0)h(i)h(j) \end{aligned}$$

16.2 Writing in terms of ∇

That double sum is pretty unwieldy. So now let's look at how we can write this using the ∇ notation. First we need the transpose of a matrix.

Definition 70
For an m by n matrix A, the **transpose** of A, written A^T is the matrix whose (i,j)th entry is the (j,i)th entry of A.

Example:

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ -1 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 3 & 0 \end{pmatrix}.$$

The transpose turns columns into rows and rows into columns.

Now, typically we think of ∇ as a row vector of operations:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}. \right)$$

So ∇^T is the corresponding column vector of operations.

Recall a row vector times a column vector is just dot product:

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1y_1 + x_2y_2 + x_3y_3.$$

A column times a row however, gives a square matrix filled with products:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} = \begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & x_3y_3 \end{pmatrix}.$$

So for $n = 3$,

$$\nabla^T \nabla = \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_1 x_2} & \frac{\partial^2}{\partial x_1 x_3} \\ \frac{\partial^2}{\partial x_2 x_1} & \frac{\partial^2}{\partial x_2^2} & \frac{\partial^2}{\partial x_2 x_3} \\ \frac{\partial^2}{\partial x_3 x_1} & \frac{\partial^2}{\partial x_3 x_2} & \frac{\partial^2}{\partial x_3^2} \end{pmatrix}.$$

This is the second derivative operator for $f : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 71
The **Hessian** of $f : \mathbb{R}^n \to \mathbb{R}$ in C^2 is the n by n matrix:

$$Hf = \nabla^T \nabla f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Can use a square matrix to get double sums of the form that we need for the second order approximation.

Fact 24
Let A be an n by n matrix, and v a column vector in $\mathbb{R}^n$. Then

$$v^T A v = \sum_{i=1}^{n} \sum_{j=1}^{n} v(i) A(i,j) v(j).$$

Definition 72
For a function $f : \mathbb{R}^n \to \mathbb{R}^n$ in C^2, the second order Taylor approximation is

$$f_2(v_0 + h) = f(v_0) + \nabla f(v_0) h + (1/2) h^T Hf(v_0) h$$

which can also be written as

$$f_2(v) = f(v_0) + \nabla f(v_0)(v - v_0) + (1/2)(v - v_0)^T Hf(v_0)(v - v_0).$$

Qotd

- Here $f(x, y) = x^3 y$:

$$\frac{\partial f}{\partial x} = 3x^2 y, \ \frac{\partial f}{\partial y} = x^3,$$

  So

$$Hf(x, y) = \begin{pmatrix} 6xy & 3x^2 \\ 3x^2 & 0 \end{pmatrix}$$

Let $v = (1,1)$, $v + tw = (x,y)$, $tw = (x-1, y-1)$. Then

$$\begin{aligned}
\nabla f(1,1) &= (3(1)^2(1), (1)^3) = (3,1) \\
Hf(1,1) &= \begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix} \\
f_2(x,y) &= f(1,1) + (3,1)\cdot(x-1,y-1) + (1/2)(x-1,y-1)\begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} \\
&= 1 + 3(x-1) + (y-1) + (1/2)(x-1,y-1)\begin{pmatrix} 6x+3y-9 \\ 3x-3 \end{pmatrix} \\
&= 3x + y - 3 + (1/2)[(x-1)(6x+3y-9) + (y-1)(3x-3)] \\
&= 3x + y - 3 + (1/2)[6x^2 + 3xy - 9x - 6x - 3y + 9 + 3xy - 3y - 3x + 3] \\
&= 3x^2 + 3xy - 6x - 2y + 3
\end{aligned}$$

- Final result is quadratic in x and y: only contains terms of degree 2, 1, or 0.
- Let's check answer. f_2 should match first two derivatives of f at $(1,1)$.

$$\begin{aligned}
\nabla f_2|_{(1,1)} &= (6x+3y-6, 3x-2)_{(1,1)} = (3,1) \\
Hf_2(x,y) &= \begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix}
\end{aligned}$$

- Note that because f_2 is a quadratic form, Hf_2 is constant
  Recall: for $f(x) = ax^2 + bx + c$, $f''(x) = 2a$ is constant

Testing the approximation for qotd

- Consider $f(1.01, 0.98) = 1.00969498$. This has linear approximation of $f(1.01, 0.98) \approx 1 + 3(0.01) + 1(-0.02) = 1.01$
- Quadratic approximation: $f(1.01, 0.98) \approx 1 + 3(0.01) + 1(-0.02) + (1/2)(6(0.01)^2 + 6(0.01)(-0.02) + 0(-0.02)^2) = 1.00939498$.

Problems

16.1: Let $f(x,y) = \sin(x+2y)$. Find the Hessian of f.

16.2: Continuing the last problem, find the second order Taylor approximation to f around the point $(\pi/2, 0)$.

16.3: Suppose $f(x,y) = \cos(x+2y)$.

(a) Find the gradient of f.
(b) Find the Hessian of f.
(c) What is $f_2(x,y)$, the second order approximation to f at $(x,y) = (0,0)$?
(d) In what direction should one move from $(\tau/4, \tau/4)$ in order to increase the value of f as quickly as possible?

17 Hessians and Maxima/Minima

Question of the Day Find all critical pts of

$$f(x,y) = e^{-(x^2+y^2)},$$

and determine if they are local maxima, local minima, or saddle points.

Today

- Using the second derivative to find local optima

Recall

- For $f \in C^2$, f has the following second order approximation:

$$f_2(x_0 + h) = f(x_0) + f'(x_0)h + (1/2)hf''(x_0)h$$
$$f_2(x) = f(x_0) + f'(x_0)(x - x_0) + (1/2)(x - x_0)f''(x_0)(x - x_0).$$

  [Note $h = x - x_0$ so $x_0 + h = x$ here.]

- If x_0 is a critical point, $f'(x_0) = 0$, so

$$f_2(x_0 + h) = f(x_0) + (1/2)h^2 f''(x_0).$$

- For $h \neq 0$ (either positive or negative), $h^2 > 0$.

$$f_2(x_0 + h) > f(x_0) \Leftrightarrow f''(x_0) > 0$$
$$f_2(x_0 + h) < f(x_0) \Leftrightarrow f''(x_0) < 0$$

Fact 25
Let $f : \mathbb{R} \to \mathbb{R}$ have $f \in C^2$. Then if $f'(x_0) = 0$ and $f''(x_0) > 0$, then $(x_0, f(x_0))$ is a local minimum. If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $(x_0, f(x_0))$ is a local maximum.

- Note, if $f'(x_0) = 0$ and $f''(x_0) = 0$ could be local maximum, minimum, or neither!

$$f(x) = x^4 \text{ (local min)}, \; f(x) = -x^4 \text{ (local max)}, \; f(x) = x^3 \text{ (neither)}.$$

17.1 Positive and Negative Definite matrices

Higher dimensions

- Multivariate 2nd order (quadratic) approximation

$$f(v_0 + h) = f(v_0) + \nabla f(v_0) \cdot h + (1/2)h^T Hf(v_0)h.$$

- If v_0 is a critical point then $\nabla f(v_0)$ is zero vector.

$$f_2(v_0 + h) = f(v_0) + (1/2)h^T Hf(v_0)h.$$

- To find if $(v_0, f(v_0))$ is a local min or max, need to know, is $h^T Hf(v)h$ positive or negative?

Definition 73
Let A be a symmetric n by n matrix.

- $(\forall w \neq 0)(w^T Aw > 0)$, call A **positive definite**
- $(\forall w \neq 0)(w^T Aw < 0)$, call A **negative definite**
- $(\forall w \neq 0)(w^T Aw \geq 0)$, call A **nonnegative definite**
- $(\forall w \neq 0)(w^T Aw \leq 0)$, call A **nonpositive definite**

Fact 26
Let v be a critical point. If $Hf(v)$ is positive definite, then v is a local minima. If $Hf(v)$ is negative definite, then v is a local maxima.

Definition 74
Let v be a critical point for $f \in C^2$. Then if v is neither a local maximum nor a local minimum, then call v a **saddle point**.

Fact 27
A is negative definite iff $-A$ is positive definite

Because of this, we only need a test to determine if a matrix is positive definite. If you take a linear algebra course, you will learn more general tests, but for this course we will just do the 2×2 case.

17.2 Determining if a 2×2 matrix is positive definite

- Now consider how we can tell if a 2×2 matrix is positive definite. First consider the general case:

$$\begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} ah_1 + bh_2 \\ bh_1 + ch_2 \end{pmatrix} = ah_1^2 + 2bh_1h_2 + ch_2^2$$

- The expression $ah_1^2 + 2bh_1h_2 + ch_2^2$ is called a *quadratic form.*
- Dividing through by h_2^2 does not change the sign:

$$a(h_1/h_2)^2 + 2b(h_1/h_2) + c = aw^2 + 2bw + c, \; w = h_1/h_2.$$

- Can tell if $aw^2 + bw + c$ is always positive, always negative, or sometimes positive and sometimes negative by determining if a and the *discriminant* of the equation are positive or negative.

Fact 28
The quadratic equation $aw^2 + bw + c$ is always positive when $a > 0$ and $b^2 - 4ac < 0$, always negative when $a < 0$ and $b^2 - 4ac < 0$, and can be zero when $b^2 - 4ac \geq 0$.

From the earlier discussion this gives:

Fact 29
A matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive definite if $a > 0$ and $b^2 - ac < 0$, and is negative definite if $a < 0$ and $b^2 - ac < 0$.

Later on we will set the *determinant* of the 2×2 matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ to be $ac - b^2$. Using this terminology:

Fact 30
A matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive definite if $a > 0$ and it has positive determinant.

Fact 31
If a 2×2 Hessian matrix has $b^2 - ac > 0$ at a critical point, then it is a saddle point.

QotD

- $f(x, y) = \exp(-(x^2 + y)^2)$, so

$$\frac{\partial f}{\partial x} = -2x\exp(-(x^2 + y^2)),\ \frac{\partial f}{\partial y} = -2y\exp(-(x^2 + y^2)).$$

  Since $\exp(-(x^2 + y^2)) > 0$ for all x and y, $\nabla f(x, y) = (0, 0) \Rightarrow (x, y) = (0, 0)$.

- To get Hessian, need second partials:

$$\frac{\partial^2 f}{\partial x^2} = [-2 + 4x^2]\exp(-(x^2 + y^2)) \qquad \frac{\partial^2 f}{\partial y \partial x} = 4xy\exp(-(x^2 + y^2))$$
$$\frac{\partial^2 f}{\partial x \partial y} = 4xy\exp(-(x^2 + y^2)) \qquad \frac{\partial^2 f}{\partial y^2} = [-2 + 4y^2]\exp(-(x^2 + y^2))$$

  So $Hf(0, 0)$ is

$$Hf(0, 0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

  Here

$$a = -2 < 0,\ b^2 - 4ac = 0 - (4)(-2)(-2) = -16 < 0,$$

  so Hessian is negative definite, and $(0, 0)$ is a $\boxed{\text{local max}}$.

Monkey saddle

- A *monkey saddle* has three downward directions, and three upward, so a monkey can straddle it with both legs and a tail.
- An example is $f(x, y) = x(x^2 - 3y^2)$.

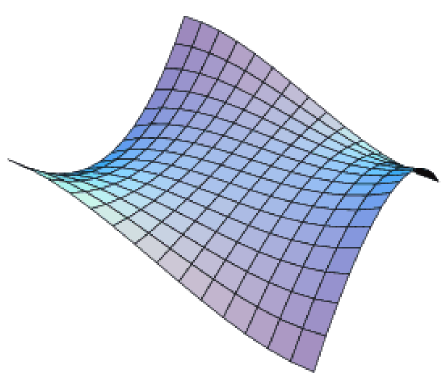


Figure 2: Taken from mathworld.wolfram.com/MonkeySaddle.html on 14 Sept, 2015.

- $$\nabla f = (2x^2 - 3y^2, -6xy),$$

  So the only critical point has $-6xy = 0$. If $x = 0$, $-3y^2 = 0 \Rightarrow y = 0$, and if $y = 0$, then $2x^2 = 0 \Rightarrow x = 0$, so either way $(0,0)$ is the only critical point.

- The Hessian is:
  $$Hf(x,y) = \begin{pmatrix} 4x & -6y \\ -6y & -6x \end{pmatrix}, \; Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- Our facts don't tell us if local max, local min, or saddle point!

- Suppose $x = y$. Then $f(x,x) = x(x^2 - 3x^2) = -2x^3$. This is positive for $x < 0$ and negative for $x > 0$, so x cannot be either a local maximum or local minimum.

Problems

17.1: Are the following matrices positive definite, negative definite, or neither?

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 3 \\ 3 & 2 \end{pmatrix}$

18 Integrals of functions of more than one variable

Question of the Day Find
$$\int_{(x,y)\in[0,1]\times[2,3]} x^2 + y^2 \; d\mathbb{R}^2.$$

Today

- Iterated integrals
- Integrals over two dimensional regions

The question of the day is an example of an integral over a two dimensional space, in this case $\mathbb{R}^2$. In order to solve this, we would like to turn the problem into a sequence of integrals over one dimensional space. That is, we would like it if:

$$\int_{(x,y)\in[0,1]\times[2,3]} x^2 + y^2 \; d\mathbb{R}^2 = \int_{x\in[0,1]} \left[\int_{y\in[2,3]} x^2 + y^2 \; dy\right] \; dx.$$

We can do the interior integral with respect to y, treating x as a constant (similar to when we did partial derivatives):

$$\begin{aligned}\int_{y\in[0,1]} x^2 + y^2 \; dy &= x^2 y + y^3/3|_2^3 \\ &= 3x^2 + 3^3/3 - (2x^2 + 2^3/3) \\ &= x^2 + 19/3\end{aligned}$$

Then we can do the outer integral:

$$\begin{aligned}\int_{(x,y)\in[0,1]\times[2,3]} x^2 + y^2 \; d\mathbb{R}^2 &= \int_{x\in[0,1]} x^2 + 19/3 \; dx \\ &= x^3/3 + (19/3)x|_0^1 \\ &= 1/3 + 19/3 \\ &= 20/3 \approx \boxed{6.666}\end{aligned}$$

When we write an integral as a sequence of nested one dimensional integrals, we call it *iterated integrals.* So when can we turn a single integral into iterated integrals? There are two main theorems that tell us when this is possible.

Theorem 5 (Fubini and Tonelli)
Suppose $A \subseteq \mathbb{R}^2$ and we wish to calculuate

$$I = \int_{(x,y)\in A} f(x,y) \; d\mathbb{R}^2.$$

Suppose one of the following conditions holds:

1: Tonelli: $f(x,y) \geq 0$ for all $(x,y) \in A$.

2: Fubini: $\int_{(x,y)\in A} |f(x,y)| \; d\mathbb{R}^2 < \infty$.

Then

$$I = \int_{\{x|(\exists y)((x,y)\in A)\}} \left[\int_{\{y|(x,y)\in A\}} f(x,y) \; dy \; dx\right] = \int_{\{y|(\exists x)((x,y)\in A)\}} \left[\int_{\{x|(x,y)\in A\}} f(x,y) \; dx \; dy\right]$$

18.1 Fubini's Theorem (bounded Version)

Now consider the problem of how we can verify the Fubini condition without going to the work of calculating the integral of $|f|$ over A. Suppose that f is bounded over A. Then the integral of $|f|$ over A is finite.

Definition 75
Say that function $f : A \to \mathbb{R}$ is **bounded** if

$$(\exists M)(\forall v \in A)(|f(v)| \leq M).$$

Fact 32
For $I = \int_{(x,y)\in A} f(x, y)\ d\mathbb{R}^2$, suppose

$$(\forall(x, y) \in A)(|f(x, y)| \leq M).$$

Then $I \leq M \cdot \text{area}(M)$.

So if f is a bounded function, and $\text{area}(M)$ is known to be finite, then we automatically know that Fubini applies!

Lemma 1 (Compact Fubini)
Let $[a, b]$ and $[c, d]$ be closed, bounded intervals. If the function f is bounded over $[a, b] \times [c, d]$, then

$$\begin{aligned}\int_{(x,y)\in[a,b]\times[c,d]} f(x, y)\ d\mathbb{R}^2 &= \int_{x\in[a,b]} \int_{y\in[c,d]} f(x, y)\ dy\ dx \\ &= \int_{y\in[c,d]} \int_{x\in[a,b]} f(x, y)\ dx\ dy.\end{aligned}$$

Remember the extreme value theorem gives that if f is continuous, then such an M always exists, so Fubini applies.

Fact 33
If f is continuous over $[a, b] \times [c, d]$, then $(\exists M)(\forall v \in A)(|f(v)| \leq M)$.

18.2 Using the Tonelli condition

Unbounded regions

- Consider the following integral:

$$\int_{(x,y)\in[0,\infty)\times[0,\infty)} \exp(-x - 2y)\ d\mathbb{R}^2.$$

- Cannot apply Fubini bounded version since domain unbounded.
- Can use Tonelli!
- Both Tonelli and Fubini apply to higher dimensional spaces, $\mathbb{R}^n$, not just $\mathbb{R}^2$.
- In $\mathbb{R}^n$, get iterated integral nested n layers deep.

Example Suppose that we want to find

$$I = \int_{(x,y)\in[0,\infty)\times[0,\infty)} \exp(-x - 2y)\ d\mathbb{R}^2.$$

- First check nonnegativity: $\exp(-x-2y) \geq 0$ for all $(x, y) \in [0, \infty) \times [0, \infty)$
- Use Tonelli to break it into an iterated integral:

$$I = \int_{x\in[0,\infty)} \int_{y\in[0,\infty)} \exp(-x-2y)\ dy\ dx.$$

- Working inside out, solve iterated integral:

$$\begin{aligned} I &= \int_{x\in[0,\infty)} \frac{\exp(-x-2y)}{-2}\Big|_0^\infty\ dx \\ &= \int_{x\in[0,\infty)} \exp(-x)/2\ dx \\ &= \int_{x\in[0,\infty)} \exp(-x)/2\ dx \\ &= \exp(-x)/(-2)|_0^\infty \\ &= 1/2 = \boxed{0.5000}. \end{aligned}$$

Nonpositive functions The Tonelli condition was written for nonnegative functions, but can easily be applied to nonpositive functions using the fact that integration is a linear operator:

$$\int_A f\ dA = -\int_A -f\ dA.$$

If $f \leq 0$ then $-f \geq 0$, so Tonelli can now be applied.

Problems

18.1: State whether or not Tonelli's Theorem, compact Fubini's Theorem, both, or neither apply to the following integrals.

(a) $\int_{(x,y)\in\mathbb{R}^2} x^2 + y^2\ d\mathbb{R}^2$.

(b) $\int_{(x,y)\in[0,2]\times[-1,1]} x + y\ d\mathbb{R}^2$.

(c) $\int_{(x,y)\in[0,2]\times[-1,1]} x + |y|\ d\mathbb{R}^2$.

(d) $\int_{(x,y)\in[0,\infty)\times[0,\infty)} 10 - (x-2)^2 + y^2\ d\mathbb{R}^2$.

18.2: Calculate the following integral:

$$I = \int_{(x,y)\in[0,\pi/3]\times[0,2\pi/3]} \sin(x+2y)\ d\mathbb{R}^2.$$

19 More two-dimensional integrals

Question of the Day Let A be the triangle connecting $(0,0)$, $(0,1)$, and $(1,1)$. Find

$$\int_{(x,y)\in A} x - y \; d\mathbb{R}^2.$$

Today

- More integrals in higher dimensions.
- Working with shapes other than rectangles.

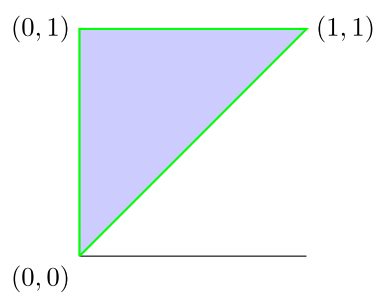
Recap

- Fubini applies when $\int_A |f| \; dA$ is finite
- Tonelli applies when $f \leq 0$ over A.
- Both allow turning n dimensional integral into n iterated integrals

Qotd Let A be the triangle connecting $(0,0)$, $(0,1)$ and $(1,1)$. Find

$$\int_{(x,y)\in A} x - y \; dA.$$

- Start by graphing the region.

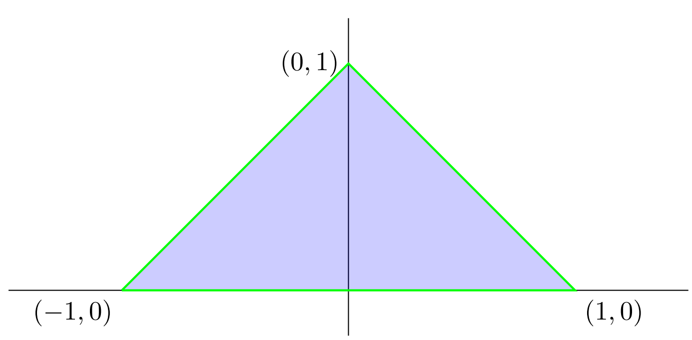


- Here the limit region is compact, and $x - y$ is continuous, so Fubini can be applied to say that the two-dimensional integral can be found using an iterated integral process.

Example Let B be the triangle connecting $(-1,0)$, $(0,1)$ and $(1,0)$. Find

$$\int_{(x,y)\in B} x^2 y \; dB.$$

- First let's graph the region:



- Because x^2y is continuous and A is bounded, Fubini's theorem applies and we can use an iterated integral. Suppose we do x on the outside. Then the smallest x can be is -1, and the largest x can be is 1. So our iterated integral will have the form:

$$I = \int_{x=-1}^{1} \int_{y \in ???} x^2 y \; dy \; dx.$$

- Now let's figure out the bounds on the y. For the triangle, there are three inequalities:

$$\begin{aligned} y &\geq 0 \\ y &\leq x + 1 \\ y &\leq -x + 1. \end{aligned}$$

The $y \geq 0$ gives the lower bound. For $y \leq x+1$ and $y \leq -x+1$, we can say that $y = \min\{x+1, -x+1\}$. So the iterated integral becomes:

$$I = \int_{x=-1}^{1} \int_{y=0}^{\min\{x+1,-x+1\}} x^2 y \; dy \; dx.$$

- That minimum in the limit is going to cause us trouble. So let's get rid of it. When $x \in [-1, 0]$, then $\min\{x+1, -x+1\} = x+1$. When $x \in [0, 1]$, then $\min\{x+1, -x+1\} = -x+1$. So break up the outer integral into these two pieces, so

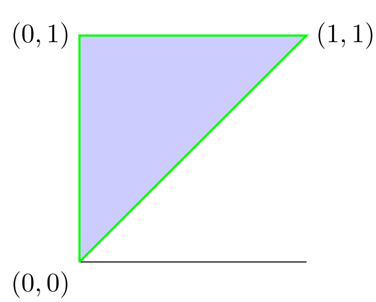
$$\begin{aligned} I &= \int_{x=-1}^{0} \int_{y=0}^{x+1} x^2 y \; dy \; dx + \int_{x=0}^{1} \int_{y=0}^{-x+1} x^2 y \; dy \; dx \\ &= \int_{x=-1}^{0} x^2 y^2/2|_0^{x+1} \; dx + \int_{x=0}^{1} x^2 y^2/2|_0^{-x+1} \; dx \\ &= \int_{x=-1}^{0} x^2(x+1)^2/2 \; dx + \int_{x=0}^{1} x^2(-x+1)^2/2 \; dx \\ &= \int_{x=-1}^{0} (1/2)(x^4 + 2x^3 + x^2) \; dx + \int_{x=0}^{1} (1/2)(x^4 - 2x^3 + x^2) \; dx \\ &= (1/2)[(1/5)x^5 + (2/4)x^4 + (1/3)(x^3)]|_{-1}^{0} + (1/2)[(1/5)x^5 - (2/4)x^4 + (1/3)(x^3)]|_0^1 \\ &= (1/2)[1/5 - 1/2 + 1/3 + 1/5 - 1/2 + 1/3] = 1/30 \approx \boxed{0.03333}. \end{aligned}$$

Qotd

- Goal, find $I = \int_{(x,y) \in A} x - y \; d\mathbb{R}^2$
- $x - y$ is continuous, $[0, 1] \times [0, 1]$ compact, use Fubini:

$$\begin{aligned} I &= \int_{(x,y) \in [0,1] \times [0,1]} (x - y)\mathbb{1}((x, y) \in A) \; d\mathbb{R}^2 \\ &= \int_{x \in [0,1]} \int_{y \in [0,1]} (x - y)\mathbb{1}((x, y) \in A) \; dy \; dx. \end{aligned}$$

- Tricky part: use the indicator function to rewrite the limits of the inside part. Good idea to first draw picture of region.



- Now look at the equations describing the triangle. The three inequalities are:

$$x \geq 0, \; y \leq 1, y \geq x.$$

- Construct limits from the inside out. So for $\int_x \int_y$ [stuff] $dy\,dx$, first do y, then x. Since the y triangle is inside the x triangle, the limits of y can be functions of x. In this case, given the value of x, $y \in [x, 1]$. Then there are x values that run from 0 up to 1. Hence

$$\begin{aligned} I &= \int_{x\in[0,1]} \int_{y\in[0,1]} (x-y)\mathbb{1}((x,y) \in A)\; dy\; dx \\ &= \int_{x\in[0,1]} \int_{y\in[x,1]} (x-y)\; dy\; dx. \end{aligned}$$

- Now do the integrations:

$$\begin{aligned} I &= \int_{x\in[0,1]} xy - y^2/2|_{y=x}^1 \; dx \\ &= \int_{x\in[0,1]} (x - 1/2) - (x^2 - x^2/2)\; dx \\ &= \int_{x\in[0,1]} x - 1/2 - x^2/2 \; dx \\ &= x^2/2 - x/2 - x^3/6|_0^1 = -1/6 \approx \boxed{\text{-0.1666}} \end{aligned}$$

19.1 When Fubini and Tonelli fail

- How can we handle positive and negative functions over unbounded limits?
- Fubini's (bounded version) only bounded limits, Tonelli's only nonnegative functions.
- Use more general Fubini's Theorem.

Fact 34
Suppose

$$I_+ = \int_{v\in A: f(v)\geq 0} f(v)\; d\mathbb{R}^n \text{ and } I_- = \int_{v \in A: f(v)\leq 0} -f(v)\; d\mathbb{R}^n$$

are both finite. Then

$$I = \int_{v\in A} f(v)\; d\mathbb{R}^n = I_+ - I_-$$

- Note that this also comes up in finding integrals of absolute values of functions:

$$\int_{v\in A} |f(v)|\; d\mathbb{R}^n = I_+ + I_-.$$

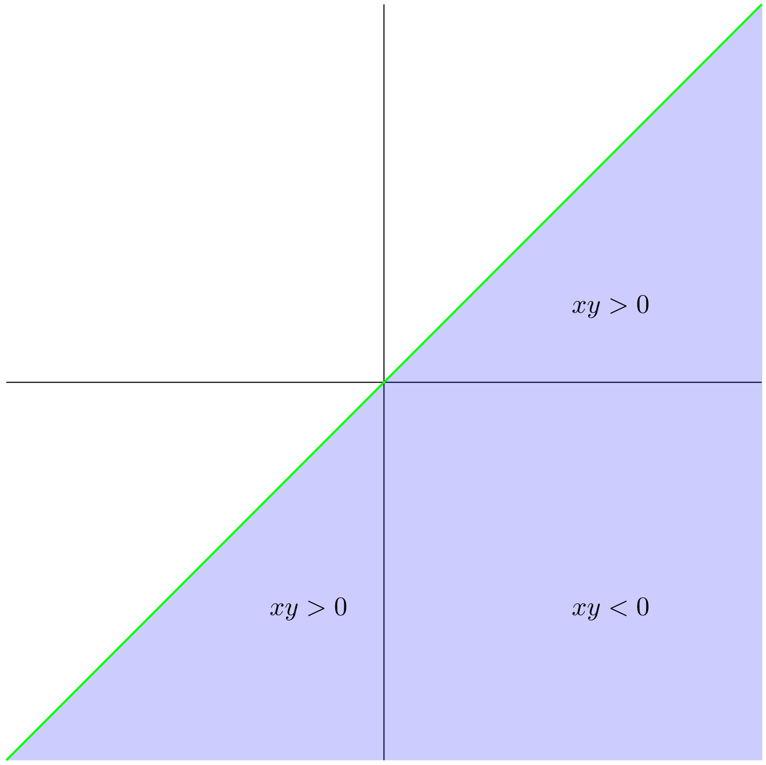
- Can always use Tonelli to find I_+ and I_- since the integrand is always nonnegative in region.

Example

- What is

$$I = \int_{x\geq y} xy \exp(-(x^2+y^2))\; d\mathbb{R}^2?$$

- Here $\exp(-(x^2+y^2)) > 0$, so break into places where $xy > 0$ and $xy < 0$, tackle separately with Tonelli.



$$
\begin{aligned}
I &= I_1 - I_2 + I_3 \\
I_1 &= \int_{x=0}^{\infty} \int_{y=0}^{x} f(x,y)\ d\mathbb{R}^2 \\
I_2 &= \int_{x=0}^{\infty} \int_{y=-\infty}^{0} -f(x,y)\ d\mathbb{R}^2 \\
I_3 &= \int_{x=-\infty}^{0} \int_{y=-\infty}^{x} f(x,y)\ d\mathbb{R}^2 \\
I &= (1/8) - (1/4) + (1/8) = 0.
\end{aligned}
$$

Problems

19.1: Let B be the region strictly inside the triangle connecting the points $(0,0)$, $(1,0)$, and $(1,1)$. Find

$$\int_B x^{-3/2}\ d\mathbb{R}^2$$

or show that it does not converge.

19.2: Find

$$\int_{y \le x+2} \frac{xy}{(1+x^2)(1+y^2)}\ d\mathbb{R}^2,$$

or show that it does not converge.

20 Riemann integration in higher dimensions

Question of the Day What is the Riemann integral of $f : \mathbb{R}^n \to \mathbb{R}$ over $A \subset \mathbb{R}^n$?

Today

- How the Riemann integral is formally defined in n dimensions.

20.1 Riemann length and area

Start with the length of an interval:

Definition 76
The Riemann length of an open interval (a, b) or a closed interval $[a, b]$ is $b - a$.

Next we define the area of a rectangle.

Definition 77
The Riemann area of the open rectangle $(a, b) \times (c, d)$ or the closed rectangle $[a, b] \times [c, d]$ is $(d - c)(b - a)$.

Definition 78
A region A is **simple** if it is a collection of a finite number of rectangles, so has the form

$$A = \cup_{i=1}^{n} A_i,$$

where each A_i is a rectangle of the form $(a_i, b_i) \times (c_i, d_i)$ or $[a_i, b_i] \times [c_i, d_i]$. Let $r(A) = \sum_{i=1}^{n} \text{area}(A_i) = \sum_{i=1}^{n} (d_i - c_i)(b_i - a_i)$.

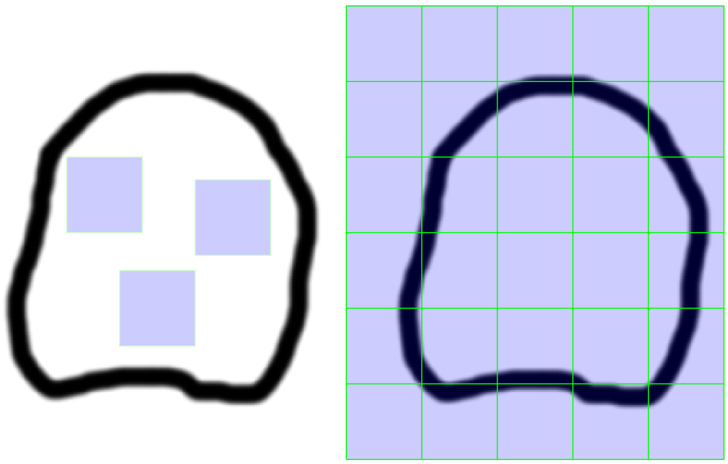
Definition 79
Suppose $R = \cup R_i$. Then R is **disjoint** if none of the R_i overlap, so so $(\forall i \neq j)(R_i \cap R_j = \emptyset)$.

The area of a disjoint simple set is just the sum of the area of the rectangles in the set.

Fact 35
If A is simple and disjoint, then $\text{area}(A) = r(A)$.

To define areas for more complicated regions, begin with the idea that if $A \subseteq B$, where A is disjoint, then $\text{area}(A) = r(A) \leq \text{area}(B)$. On the other hand if $B \subseteq A$, where A is simple, then $\text{area}(B) \leq r(A)$.

- So for every $A \subseteq B$ where A simple and disjoint, $r(A)$ is a lower bound for a possible value of $\text{area}(A)$.
- And for every A where A simple and $B \subseteq A$, $r(A)$ is an upper bound for a possible value of $\text{area}(A)$.
- If there is exactly one number at least as big as all the lower bounds, and as small or smaller than all the upper bounds, then that is the Riemann area.



Riemann area $\in [3, 15]$

Definition 80
Suppose that B is a bounded region. Let $\mathcal{S}_1$ denote the set of simple disjoint regions in $\mathbb{R}^2$, and $\mathcal{S}_2$ denote the set of simple regions. Suppose there is a single number s such that

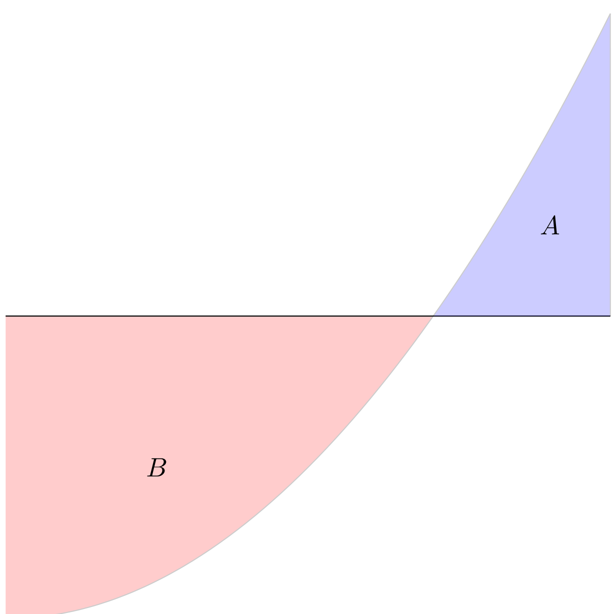
$$(\forall A \in \mathcal{S}_1 : A \subseteq B)(r(A) \leq s),$$

and

$$(\forall A \in \mathcal{S}_2 : B \subseteq A)(s \leq r(A)).$$

Then the **Riemann area** of the region B is s.

Definition 81
For f a function over $[a, b]$, let $A = \{(x, y) : x \in [a, b], 0 \leq y \leq f(x)\}$, and $B = \{(x, y) : x \in [a, b], f(x) \leq y \leq 0\}$. Then the **Riemann integral** of f from a to b is the Riemann area of A minus the Riemann area of B if both these exist.



20.2 Riemann hypervolume

Now expand the definitions to n dimensional space.

Definition 82
The **Riemann hypervolume** of $[a_1, b_1] \times \cdots [a_n, b_n]$ (or $(a_1, b_1) \times \cdots \times (a_n, b_n)$) is $(b_1 - a_1) \cdots (b_n - a_n)$.

Definition 83
A region in $\mathbb{R}^n$ is **simple** if it consists of the union of a finite number of boxes of the form $[a_1, b_1] \times \cdots \times [a_n, b_n]$ or $(a_1, b_1) \times \cdots \times (a_n, b_n)$. For such a region, let $r(A)$ be the sum of the Riemann hypervolumes of the boxes comprising the region.

Definition 84
Suppose that B is a bounded region. Let $\mathcal{S}_1$ denote the set of simple disjoint regions in $\mathbb{R}^n$, and $\mathcal{S}_2$ denote the set of simple regions. Suppose there is a value s such that

$$(\forall A \in \mathcal{S}_1 : A \subseteq B)(r(A) \leq s),$$

and

$$(\forall A \in \mathcal{S}_2 : A \subseteq B)(s \leq r(A)),$$

Then call s **Riemann hypervolume** of the region B.

The Riemann integral is then the hypervolume under the function where it is nonnegative minus the hypervolume above the function where it is positive.

Definition 85
For f a function over bounded $R \subset \mathbb{R}^n$, let

$$A = \{(x_1, \ldots, x_n, y) : (x_1, \ldots, x_n) \in R, 0 \leq y \leq f(x_1, \ldots, x_n)\}, \text{ and}$$
$$B = \{(x_1, \ldots, x_n, y) : (x_1, \ldots, x_n) \in R, f(x_1, \ldots, x_n) \leq y \leq 0\}.$$

Then the **Riemann integral** of f over R is the Riemann area of A minus the Riemann area of B if both these exist.

Problems

20.1: Find $r(A)$ for the following examples:

(a) $A = (0, 3) \times (4, 8)$.
(b) $A = ((0, 3) \times (4, 8)) \cup ((-1, 1) \times (-2, 2))$.
(c) $A = (0, 3) \times (4, 8) \times (10, 12)$.
(d) $A = [0, 3] \times [4, 8] \times [10, 12]$.

21 Solving integrals with polar coordinates

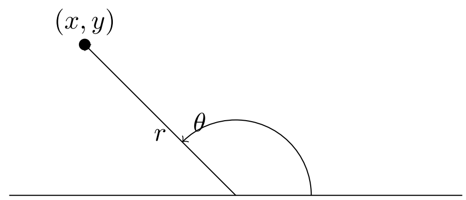
Question of the Day What is $\int_{(x,y):x^2+y^2\leq 1} e^{-(x^2+y^2)}\ d\mathbb{R}^2$?

Today

- Converting integrals between rectangular (Cartesian) and polar coordinates.

21.1 Polar coordinates

- Coordinate systems are different ways of recording points in space
- Rectangular use horizonal and vertical distance
- Polar coordinates use angle and distance from origin



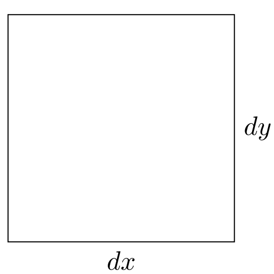
- To convert back and forth, use trigonometry:

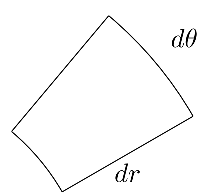
$$x = r\cos(\theta) \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin(\theta) \qquad \theta = \arctan(y/x)$$

- Often, easier to describe rotationally symmetric regions using polar:

$$\{(x, y) : x^2 + y^2 \leq 1\} = \{r \leq 1, \theta \in [0, 2\pi]\}$$

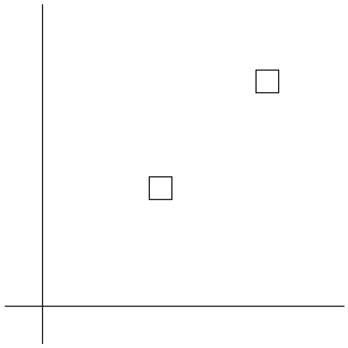
- Differentials for the two regions are different



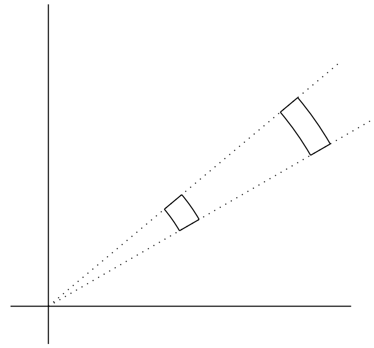


Area $= dx \cdot dy$ Area $\neq dr \cdot d\theta$

- Note: the square doesn't change area as you move towards and away from the origin:



Area = $dx\ dy$



Area = $r\ dr\ d\theta$

- Why is area of differential polar "crust" $r\ dr\ d\theta$?
- Area of pizza of radius r: πr^2.
- Area of slice of pizza: $\theta r^2/2$.
- Area of differential crust:

$$\begin{aligned}(1/2)d\theta(r+dr)^2 - (1/2)d\theta r^2 &= (1/2)d\theta(r + 2rdr + (dr)^2) - (1/2)d\theta r^2 \\ &= (1/2)d\theta(2rdr + (dr)^2) \\ &= r\ dr\ d\theta.\end{aligned}$$

Fact 36
For polar coordinates,

$$dx\ dy = r\ dr\ d\theta.$$

21.2 Limits, Integrand, Differential

- When changing from rectanglar to polar ((x, y) to (r, θ)) have to change
  1: Limits: write limits in terms of (r, θ)
  2: Integrand: put r and θ into x and y
  3: Differential: Change $dx\ dy$ to $r\ dr\ d\theta$

Qotd

- Limits:

$$\int_{(x,y):x^2+y^2\leq 1} \exp(-(x^2+y^2))\ dx\ dy = \int_{r\leq 1} \exp(-(x^2+y^2))\ dx\ dy$$

- Integrand:

$$I = \int_{r\leq 1} \exp(-r^2)\ dx\ dy$$

- Differential:

$$I = \int_{r\leq 1} \exp(-r^2) r\ dr\ d\theta.$$

- Now use Tonelli (note r always nonnegative)

-
$$
\begin{aligned}
I &= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r \exp(-r^2)\ dr \\
&= 2\pi \int_{r=0}^{1} r \exp(-r^2)\ dr \\
&= 2\pi \exp(-r^2)/(-2)|_0^1 \\
&= 2\pi[-(1/2)\exp(-1) - (-1/2)\exp(0)] \\
&= \pi(1 - 1/e) \approx \boxed{1.985}
\end{aligned}
$$

- This is a special case of a very general framework for moving from one set of coordinates to another.
- Much of what we do in the rest of the course will be changing integrals over differentials from one coordinate system to another in order to make problems easier.

Problems

21.1: Find

$$
\int_{x^2+y^2 \leq 9} \frac{1}{(x^2+y^2)^{1/4}}\ d\mathbb{R}^2.
$$

22 Vector Fields

Question of the Day Find the integral of $F(x,y) = (x^2, x+y)$ over the parabola $x = y^2$ between $(1,-1)$ and $(1,1)$.

Today

- Vector fields

Types of functions

- Curves (input real, output vector): $C : \mathbb{R} \to \mathbb{R}^n$
- Real Valued Function (input vector, output real): $\mathbb{R}^n \to \mathbb{R}$
- Vector Fields (Input is vector, get back another vector)

$$F : \mathbb{R}^n \to \mathbb{R}^m.$$

Examples

- Wind velocity at a position
- Change in economic indicators at a given point in dataspace.
- General form for $F : \mathbb{R}^n \to \mathbb{R}^m$,

$$F(v) = (f_1(v), f_2(v), \ldots, f_m(v)),$$

  where $v \in \mathbb{R}^n$, and f_i are all real-valued functions.

Definition 86
If each $f_i \in C^n$, then $F \in C^n$.

- The QotD asks us to integrate a vector field over a curve
- Physics: needed to find work to move an object through a vector of forces
- Econ: needed to evaluate effort to move along path in indicator space.
- Recall: for a curve C,

$$dC = C'(t)\ dt.$$

- The length of a curve uses $\|dC\| = \|C'(t)\|\ dt$ for $dt > 0$.

22.1 Definition of curve integrals

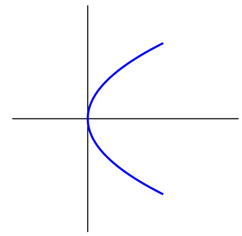
Definition 87
The **path integral** (also called **curve integral**) over the curve C for $t \in [t_0, t_1]$ of the vector field F is

$$\int_C F = \int_C F \cdot dC = \int_{t=t_0}^{t_1} F \cdot C'(t)\ dt = \int_{t=t_0}^{t_1} F(C(t)) \cdot C'(t)\ dt.$$

- In the Q of Day,

$$\int_C (x^2, x+y) = \int_C (x^2, x+y) \cdot dC,$$

  for C the parabola $x = y^2$ from $(1,-1)$ to $(1,1)$.



- Steps

  1: Parameterize the curve. Turns out you can do this any way you want. For instance:

  $$C_1(t) = (t^2, t),\ C_2(t) = (t, \sqrt{t}), C_3(t) = (e^{2t}, e^t)$$

  all work (all have $x = y^2$). Use whichever makes integral easiest. Start with C_1,

  $$C'(t) = (2t, 1).$$

  Then $t \in [-1, 1]$, since $C_1(-1) = (1, -1)$, $C_1(1) = (1, 1)$.

  2: Next step: write $F(x, y)$ in terms of t. With C_1, $x = t^2$, $y = t$,

  $$F(C(t)) = F(t^2, t) = (t^4, t^2 + t)$$

  3: Solve the integral:

  $$\begin{aligned}\int_C F &= \int_{-1}^{1} (t^4, t^2 + t) \cdot (2t, 1)\ dt \\ &= \int_{-1}^{1} 2t^5 + t^2 + t\ dt = \frac{2}{6}t^6 + \frac{1}{3}t^3 + \frac{1}{2}t^2|_{-1}^{1} \\ &= \frac{2}{6} + \frac{1}{3} + \frac{1}{2} - \left[\frac{2}{6} - \frac{1}{3} + \frac{1}{4}\right] = \frac{2}{3} \approx 0.6666.\end{aligned}$$

Intuition

- Suppose that at point (x, y), the wind is blowing with velocity $F(x, y)$. Then $\int_C F(x, y)\ dC$ is the amount of energy you get from the wind to walk along curve C
- If wind is at your back, integral is positive
- If wind is blowing you backward, integral is negative (you have to add energy to walk the path.)

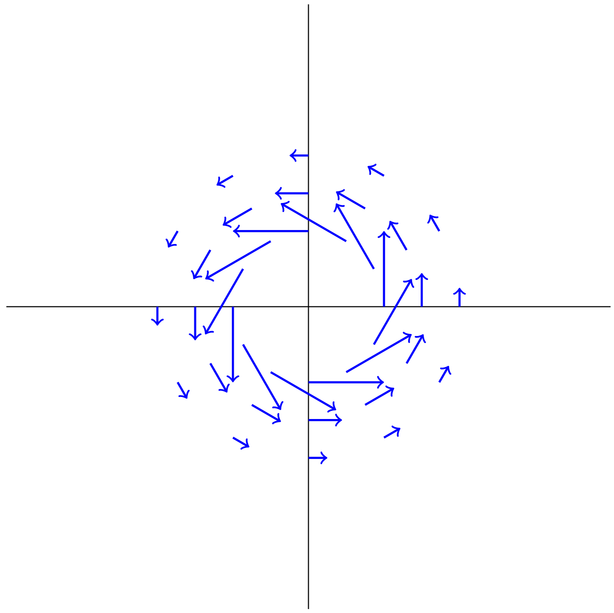
Example

- Find

  $$\int_C G,\ G(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right),$$

  where C travels one circuit counterclockwise around the circle of radius 3.

- "Wind" is moving counterclockwise: $G(x, y) = (-y/r^2, x/r^2)$



- If you move counterclockwise around circle, wind pushes you, so curve integral is positive.
- First step solution: parameterize the curve:

$$(x, y) = (3\cos(\theta), 3\sin(\theta)),$$

Second step: write G in terms of parameter θ:

$$\begin{aligned} G(C(\theta)) &= \left(\frac{-3\sin(\theta)}{9\sin^2(\theta) + 9\cos^2(\theta)}, \frac{3\cos(\theta)}{9\sin^2(\theta + 9\cos^2(\theta))} \right) \\ &= (-(1/3)\sin(\theta), (1/3)\cos(\theta)). \end{aligned}$$

Last step: solve the curve integral:

$$\begin{aligned} \int_C G &= \int_{\theta=0}^{2\pi} (-(1/3)\sin(\theta), (1/3)\cos(\theta)) \cdot (-3\sin(\theta), 3\cos(\theta)) \ dt \\ &= \int_{\theta \in [0,2\pi]} \sin^2(\theta) + \cos^2(\theta) \ d\theta \\ &= 2\pi. \end{aligned}$$

- Note: Wind weaker as get farther out, but distance traveled is greater, so overall curve integral does not depend on radius!

Two facts about 1-D integrals

1:

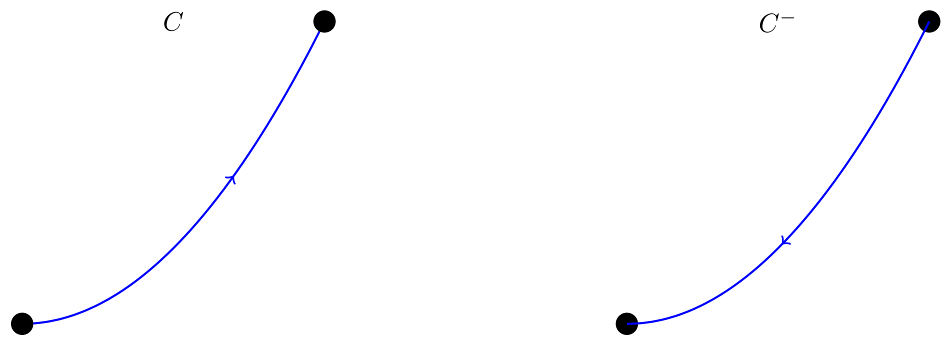
$$\int_a^b f(x) \ dx = -\int_b^a f(x) \ dx.$$

2: For $a < c < b$:

$$\int_a^b f(x) \ dx = \int_a^c f(x) \ dx + \int_c^b f(x) \ dx.$$

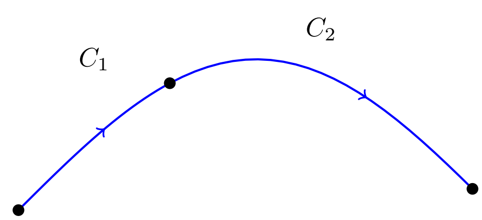
Two facts about curve integrals

1: Let C^- be the reverse curve of C:



$$\int_C F = -\int_{C^-} F.$$

2: Let $C = C_1 \cup C_2$, then



$$\int_C F = \int_{C_1} F + \int_{C_2} F.$$

23 Curve integrals for potential functions

Question of the Day Let $F(x, y, z) = (z^3y, z^3x, 3z^2xy)$, and C is a curve from $(0, 0, 0)$ to $(1, 1, 1)$. What is
$$\int_C F(x, y, z) \cdot dC?$$

Today

- Potential functions
- Generalized Fundamental Theorem of Calculus

Recall the FTC:

Theorem 6 (Fundamental Theorem of Calculus)
If $f(x) \in C^1$, then
$$\int_{x=a}^{b} f'(x)\ dx = f(b) - f(a).$$

- The FTC says if $f'(x) \in C^0$, then only value of f at boundary of $[a, b]$ matters. Boundary of $[a, b]$ is $\{a, b\}$.
- To use FTC, must find antiderivatives. For instance, $\text{ad}_z(3z^2) = z^3 + C$ for any constant C. Can use $C = 4$ for instance:
$$\int_{z=0}^{1} 3z^2\ dz = (z^3 + 4)|_0^1 = 7 - 4 = 3.$$
Usually use $C = 0$ as that is the simplest antiderivative.

23.1 The Generalized Fundamental Theorem of Calculus

Theorem 7 (Generalized Fundamental Theorem of Calculus)
Suppose $\phi : \mathbb{R}^n \to \mathbb{R} \in C^1$ and C is a curve parameterized as $C([t_0, t_1])$. Then
$$\int_C \nabla\phi \cdot dC = \phi(C(t_1)) - \phi(C(t_0)).$$

Just as with FTC, for Gen FTC, inside of curve doesn't affect integral, only first point of curve $C(t_0)$ and last point of curve $C(t_1)$.

Using the gen FTC

- In general, GFTC harder to use than FTC.
- To use for
$$\int_C (z^3y, z^3x, 3z^2xy),$$
have to find ϕ such that
$$\frac{\partial\phi}{\partial x} = z^3y,\ \frac{\partial\phi}{\partial y} = z^3x,\ \frac{\partial\phi}{\partial z} = 3z^2xy.$$

Partial antiderivatives

- Remember, $\partial A(y,z)/\partial x = 0$ for any function A of y and z. In 1-D
$$\text{ad}_x(3z^2) = z^3 + C.$$
In 3-D:
$$\text{ad}_x(3z^2xy) = z^3 + C(x,y).$$
Want to solve:
$$\text{ad}_x(z^3y) = \text{ad}_y(z^3x) = \text{ad}_Z(3z^2xy)$$
Get three equations:
$$z^3yx + A(y,z) = z^3xy + B(x,z) = z^3xy + C(x,y).$$
In this case, can make $A(y,z) = B(x,z) = C(x,y) = 0$ to make it work.
- $$\int_{C:(0,0,0) \text{ to } (1,1,1)} \nabla(z^3xy) = 1^3 \cdot 1 \cdot 1 - 0^3 \cdot 0 \cdot 0 = \boxed{1}.$$
Note answer does not depend on how the curve gets from $(0,0,0)$ to $(1,1,1)$!

23.2 Potential functions

Definition 88
Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. If $F = \nabla\phi$ for some $\phi : \mathbb{R}^n \to \mathbb{R}$, call ϕ a **potential function** for F.

Harder example

- Find the potential function that gives rise to the vector field
$$F(x,y,z) = (2xyz + y\cos(xy), x^2z + z + x\cos(xy), x^2y + y).$$
- Any such vector field has to satisfy:
$$\text{antider}_x(2xyz + y\cos(xy)) = \text{antider}_y(x^2z + z + x\cos(xy)) = \text{antider}_z(x^2y + y),$$
so find $A(y,z)$, $B(x,z)$ and $C(x,y)$ so that:
$$x^2yz + \sin(xy) + A(y,z) = x^2yz + yz + \sin(xy) + B(x,z) = x^2yz + yz + C(x,y).$$
In this case, the x^2yz cancels out, leaving
$$\sin(xy) + A(y,z) = yz + \sin(xy) + B(x,z) = yz + C(x,y).$$
Then $C(x,y) = \sin(xy)$ to complete its expression, and $A(y,z) = yz$ to complete its expression, and $B(x,z) = 0$. The final result is
$$\phi(x,y,z) = \boxed{x^2yz + yz + \sin(xy)}$$

Closed curves

Definition 89
Say $C = C([t_0, t_1])$ is a **closed curve** if $C(t_0) = C(t_1)$.

A closed curve ends where it started.

Fact 37
If F has a potential function ϕ and C is a closed curve, then $\int_C F = 0$.

Proof.
$$\int_C F = \int_{C([t_0,t_1])} \nabla\phi = \phi(C(t_1)) - \phi(C(t_0)) = 0.$$
□

How can we tell if F does not have a potential function?

- Recall that if $F = \nabla\phi$, then $F = (F_1, F_2, \ldots, F_n)$ where $F_i = D_i\phi$. So for all $i \neq j$

$$D_jF_i = D_jD_i\phi = D_iD_j\phi = D_iF_j.$$

  If one of these equations fails to hold, then F cannot have a potential function.

- Example: does $F(x, y, z) = (x + y, 2xy, 3(x^2 + z^2))$ have a potential function? No! The proof:

$$F_2(x,y,z) = 2xy \qquad D_1F_2 \qquad = 2y$$
$$D_2F_1 = 1.$$

  Since these two are not equal, no potential function can exist!

- Earlier example: $F(x, y, z) = (z^3y, z^3x, 3z^2xy)$.

$$\begin{aligned} D_1F_2 &= z^3 & D_2F_1 &= z^3 \\ D_1F_3 &= 3yz^2 & D_3F_1 &= 3yz^2 \\ D_2F_3 &= 3xz^2 & D_3F_2 &= 3xz^2. \end{aligned}$$

Fact 38
If $F = \nabla\phi$, then $D_iF_j = D_jF_i$ for all $i \neq j$. The converse is not true: it is possible for $D_iF_j = D_jF_i$ for all $i \neq j$, but still have no potential function for F.

- There are other tests for if F has a potential function, but this is the only one that we'll discuss in this course.

23.3 Proof of the Generalized Theorem of Calculus

Proof. Let C be a curve from $C(t_0)$ to $C(t_1)$. Then

$$\begin{aligned} \int_C \nabla\phi &= \int_{t=t_0}^{t_1} \nabla\phi(C(t)) \cdot C'(t)\ dt \\ &= \int_{t=t_0}^{t_1} \frac{d}{dt}\phi(C(t))\ dt && \text{[by the chain rule]} \\ &= \phi(C(t_1)) - \phi(C(t_0)) && \text{[by the FTC].} \end{aligned}$$

□

24 Changing curve integrals to integrals over area

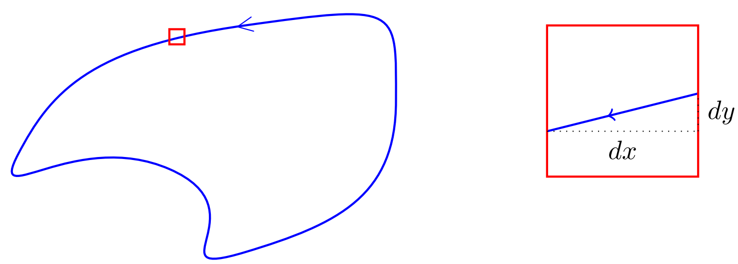
Question of the Day Find $\int_C y^2\ dx + x\ dy$ for C moving counterclockwise around a circle of radius 2 centered at the origin.

Today

- Green's Theorem

So far...

- Curve integrals tell how much energy comes from wind as you move around a path.
- $\int_C y^2\ dx + x\ dy$ says you get a little bit of energy $y^2\ dx$ when you travel a small distance dx in the x direction, and a little push of $x\ dy$ when you travel a small distance dy in the y direction.



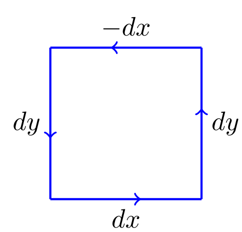
- Example: moving from $(6, 1)$ to $(6.1, 1.2)$ adds about

$$(6.1 - 6) \cdot 1^2 + (1.2 - 1) \cdot 6 \approx 1.3$$

to the integral.

- Our clever plan:

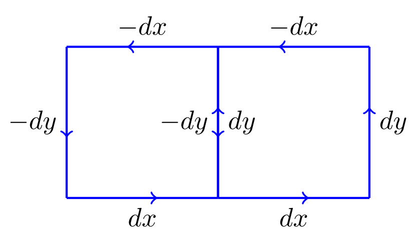
1: Divide the region into little squares



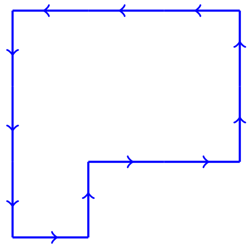
Give this curve integral around differential curve a name, for $F(x, y) = (F_1(x, y), F_2(x, y))$:

$$\int_{\text{diff. curve}} F(x, y) = I(x, y).$$

2: Put the squares next to each other. Note that the dy and $-dy$ lines cancel each other out.



3: Use enough squares to build back up to the original boundary/closed curve. [All lines inside cancel each other out]



Then

$$\int_C F = \int_{\text{area inside C}} I(x, y).$$

- So this idea converts a curve integral around C to a two dimensional integral over the area inside C.

24.1 Green's Theorem

- To use this, need to know what $I(x, y)$ is. Derive this using the linear approximations for $F = (F_1, F_2)$:

$$\begin{aligned} I(x, y) &= \int_{\text{diff. curve}} F(x, y) \\ &= \int_a F + \int_b F + \int_c F + \int_d F \\ &= \int_a F + \int_b F - \int_{c^-} F - \int_{d^-} F \end{aligned}$$

Do a and c^- first:

$$\begin{aligned} \int_a F - \int_{c^-} F &= F_1(x, y)\ dx - F_1(x, y + dy)\ dx \\ &= F_1(x, y)\ dx - (F_1(x, y) + \frac{\partial F_1}{\partial y} dy)\ dx \\ &= -\frac{\partial F_1}{\partial y}\ dx\ dy. \end{aligned}$$

- Next do b and d^-:

$$\begin{aligned} \int_b F - \int_{d^-} F &= F_2(x + dx, y)\ dy - F_1(x, y)\ dy \\ &= (F_2(x, y) + \frac{\partial F_2}{\partial x}\ dx)\ dy - F_2(x, y)\ dy \\ &= \frac{\partial F_2}{\partial x}\ dx\ dy. \end{aligned}$$

- So

$$I(x, y) = \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]\ dx\ dy$$

Definition 90
A curve has **postive orientation** if the area it encloses is on the left of the curve.

Theorem 8 (Green's Theorem)
Let $A \subset \mathbb{R}^2$ have a differentiable closed curve C as its boundary. Then if C has positive orientation, for $p(x, y), q(x, y) \in C^1$,

$$\int_C p\ dx + q\ dy = \int_A \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right)\ dy\ dx.$$

Qotd

- Here $p(x, y) = y^2$, $q(x, y) = x$.
- So

$$\begin{aligned}
\int_C y^2\ dx + x\ dy &= \int_{(x,y):x^2+y^2 \leq 2^2} (1 - 2y)\ dx\ dy \\
&= \int_{r \in [0,2]} \int_{\theta \in [0,2\pi]} r(1 - 2r\sin(\theta))\ d\theta\ dr \\
&= \int_{r \in [0,2]} r[\theta + 2r\cos(\theta)]|_0^{2\pi}\ dr \\
&= \int_{r \in [0,2]} 2\pi r\ dr \\
&= 2\pi r^2/2|_0^2 \\
&= 4\pi \approx \boxed{12.56}
\end{aligned}$$

Boundaries and integrals

- FTC: integral of f' over $[a, b]$ only depends on f at boundary of $[a, b]$
- GFTC: integral of $\nabla\phi$ over $C([a, b])$ only depends on ϕ at boundary points $\{C(a), C(b)\}$.
- Green's Theorem: integral of $\partial q/\partial x - \partial p/\partial y$ over A only depends on p and q at boundary of A.
- In general, have Stokes' Theorem:
$$\int_{\partial\Omega} \omega = \int_\Omega d\omega.$$
- The value of $d\omega$ is called a differential form, and we will define and discuss it later in the course.

25 Divergence and rotation

Question of the Day Let $F(x,y) = (y, -x)$, and C be the unit circle oriented ccw. Then for n the normal direction to the curve, show that

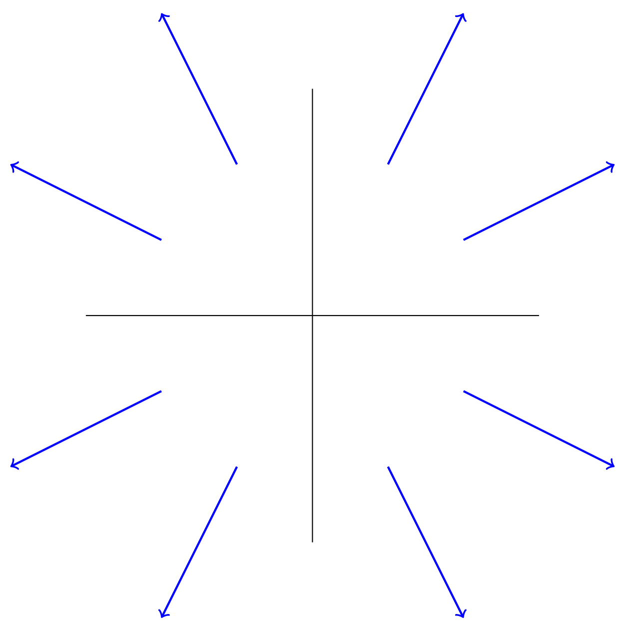
$$\int_C F \cdot dn = 0.$$

Today

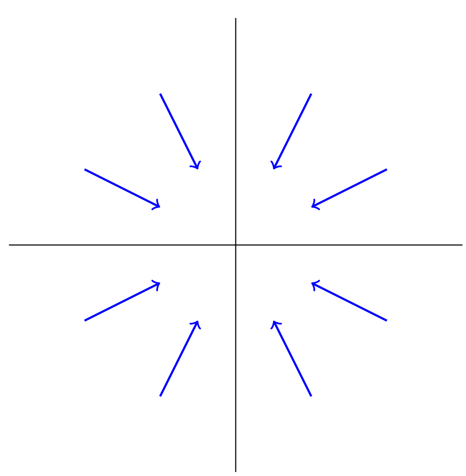
- Divergence of vector field
- Rotation of a vector field

Types of vector fields

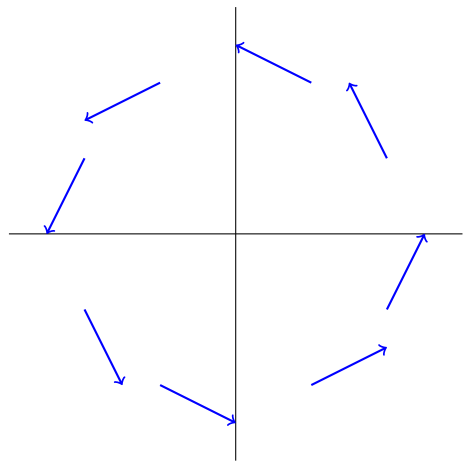
- In an exploding vector field, the wind is blowing outwards from the origin



- In a collasping vector field, wind is blowing in towards the origin.



- In rotating vector fields, objects pushed around in circles



- Divergence measures exploding/compressing, rotation measures how much spin in the vector field.

Definition 91
The **divergence** of a vector field $F = (f_1, \dots, f_n)$ is

$$\operatorname{div}(F) = \nabla \cdot F = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}.$$

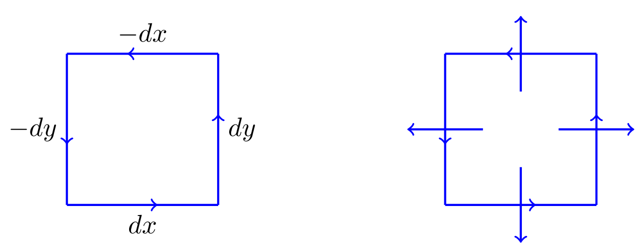
Definition 92
The **rotation** of $F = (p, q)$ is

$$\operatorname{rot}(F) = \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}.$$

- With this notation, Green's theorem becomes:

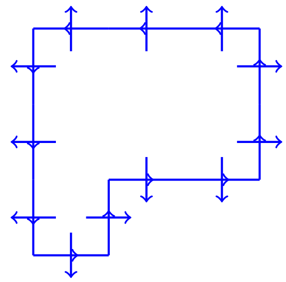
$$\int_C F = \int_{\text{inside C}} \operatorname{rot} F$$

- Rotation versus explosion:



Rotation Explosion

- For explosion, want $F \cdot n$, where n is the normal vector to the direction of motion.
- Like with Green's Theorem, summing over squares gives "explosion" past enclosing curve



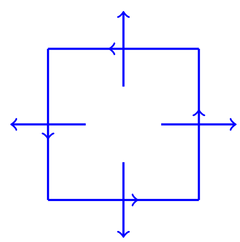
- So, letting $\square$ denote a little differential square, we have:

$$\int_{\text{boundary } A} F = \int_A \int_\square F = \int_A \text{rot}(A)\ dx\ dy \qquad \text{[Green's Theorem]}$$

$$\int_{\text{boundary } A} F \cdot n = \int_A \int_\square F \cdot dn = \int_A \text{div}(A)\ dx\ dy \qquad \text{[Divergence Theorem]}$$

$$\int_{\text{boundary } [a,b]} F = F(b) - F(a) = \int_A f'(x)\ dx \qquad \text{[FTC].}$$

- So what is $\text{div}(F)$? Consider integrating around a rectangle of width a_1 and height a_2. Then there are four pieces:



Explosion

$$I = I_1 + I_2 + I_3 + I_4$$
$$I_1 = \int_{(0,0) \text{ to } (a_1,0)} F(x,y) \cdot (0,-1)$$
$$I_2 = \int_{(a_1,0) \text{ to } (a_1,a_2)} F(x,y) \cdot (1,0)$$
$$I_3 = \int_{(a_1,a_2) \text{ to } (0,a_2)} F(x,y) \cdot (0,1)$$
$$I_4 = \int_{(0,a_2) \text{ to } (0,0)} F(x,y) \cdot (-1,0)$$

Using $F(x,y) = (f_1(x,y), f_2(x,y))$,

$$I_1 = \int_{x=0}^{a_1} -f_2(x,0)\ dx$$
$$I_2 = \int_{y=0}^{a_2} f_1(a,y)\ dy$$
$$I_3 = \int_{x=0}^{a_1} f_2(x,a)\ dx$$
$$I_4 = \int_{y=0}^{a_2} -f_1(0,a)\ dy$$

So combining I_1 and I_3 and I_2 and I_4 gives:

$$I_1 + I_3 = \int_{x=0}^{a_1} [f_2(x,a) - f_2(x,0)]\ dx$$
$$I_2 + I_4 = \int_{y=0}^{a_2} [f_2(a,y) - f_2(0,y)]\ dy$$

So now picture a_2 getting very small. Then

$$f_2(x,a_2) \approx f_2(x,0) + a_1 \frac{\partial f_2}{\partial y}(x,0)$$
$$f_2(x,a_2) - f_2(x,0) \approx a_2 \frac{\partial f_2}{\partial y}(x,0)$$

Then if a_1 is also very small, then the integrand is almost a constant, so

$$\int_{x=0}^{a_1} [f_2(x,a) - f_2(x,0)]\ dx \approx a_1 a_2 \frac{\partial f_2}{\partial y}(0,0).$$

Repeating this for y gives

$$\int_{y=0}^{a_2} [f_1(a_1,y) - f_2(0,y)]\ dy \approx a_1 a_2 \frac{\partial f_1}{\partial x}(0,0).$$

Then for $a_1 = dx$ and $a_2 = dy$,

$$I = I_1 + I_2 + I_3 + I_4 = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\ dx\ dy.$$

So

$$\text{div}(F) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = \nabla \cdot F.$$

Qotd

- For $F = (y, -x)$,
$$\text{div}(F) = \nabla \cdot F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 0 + 0 = 0.$$
- Therefore $\int_{\text{boundary } A} = \int_A \text{div}(F)\ dx\ dy = 0$.

25.1 The Divergence Theorem in 2D

If a curve is moving in direction $dC = (dx, dy)$, then the differential direction $dn = (dy, -dx)$ is perpendicular to the curve. To see this, note

$$(dx, dy) \cdot (dy, -dx) = dx\ dy - dy\ dx = 0.$$

Theorem 9 (Divergence Theorem in 2D)
Let $F(x,y) = (f_1(x,y), f_2(x,y))$, and A be a region with a closed curve as a boundary. Then

$$\int_{\text{boundary } A} F \cdot dn = \int_A \text{div}(F)\ dx\ dy = \int_A \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} \right)\ dx\ dy,$$

where the curve integral is taken to be counterclockwise.

26 Areas and volumes through integrals

Question of the Day Find the volume of the tetrahedron with vertices at $(0,0,0)$, $(0,0,1)$, $(1,0,0)$, and $(0,1,0)$.

Today

- Finding areas and volumes with integrals

Fact 39
The area of a region $A \subset \mathbb{R}^2$ is $\int_A 1\ d\mathbb{R}^2 = \int_{\mathbb{R}^2} \mathbb{1}((x,y) \in A)\ d\mathbb{R}^2$. The volume of $S \subset \mathbb{R}^3$ is $\int_S 1\ d\mathbb{R}^3 = \int_{\mathbb{R}^3} \mathbb{1}((x,y,z) \in S)\ d\mathbb{R}^3$.

- Note that since $1(v) = 1$ is a nonnegative function, can always use Tonelli to break integral into iterated integral.
- The tricky part is writing the limits of the iterated integral.
- Remember to work inside out

Qotd

$$\int_S 1\ d\mathbb{R}^3 = \int_{x\in\mathbb{R}} \int_{y\in\mathbb{R}} \int_{z\in\mathbb{R}} \mathbb{1}(\underbrace{(x,y,z) \in A}_{x\geq 0, y\geq 0, z\geq 0, x+y+z\leq 1})\ dz\ dy\ dx$$

- Unwind the indicator function: given x and y, what are permissible values of z? $z \geq 0$, and $x+y+z \leq 1$, so $z \leq 1 - x - y$. So

$$\int_S 1\ d\mathbb{R}^3 = \int_{x\in\mathbb{R}} \int_{y\in\mathbb{R}} \mathbb{1}(x \geq 0, y \geq 0, x + y \leq 1) \int_{z\in[0,1-x-y]} dz\ dy\ dx$$

- Now use the indicator to change the limits on y, then on x

$$\begin{aligned}
\int_S 1\ d\mathbb{R}^3 &= \int_{x\in\mathbb{R}} \mathbb{1}(x \geq 0, x \leq 1) \int_{y\in[0,1-x]} \int_{z\in[0,1-x-y]} dz\ dy\ dx \\
&= \int_{x\in[0,1]} \int_{y\in[0,1-x]} \int_{z\in[0,1-x-y]} dz\ dy\ dx \\
&= \int_{x\in[0,1]} \int_{y\in[0,1-x]} 1 - x - y\ dy\ dx \\
&= \int_{x\in[0,1]} (1-x)y - y^2/2|_0^{1-x} \\
&= \int_{x\in[0,1]} (1/2)(1-x)^2|_0^{1-x} \\
&= (1/2)(1/3)(1-x)^3/(-1)|_0^1 \\
&= 1/6 \approx \boxed{0.1666}.
\end{aligned}$$

26.1 Special cases of the area integral

Integrate a nonnegative function to get the area under the curve...

Fact 40
Let $A = \{(x,y) : x \in [a,b], y \in [0, f(x)]\}$ for $f(x) \geq 0$. Then the area of A is

$$\int_{x=a}^{b} f(x)\ dx.$$

Proof. We know

$$\int_{\mathbb{R}^2} \mathbb{1}((x,y) \in A)\ d\mathbb{R}^2 = \int_{x\in\mathbb{R}} \int_{y\in\mathbb{R}} \mathbb{1}(x \in [a,b], y \in [0, f(x)])\ dy\ dx$$
$$= \int_{x\in\mathbb{R}} \int (x \in [a,b]) \int_{y=0}^{f(x)}\ dy\ dx$$
$$= \int_{x=a}^{b} f(x)\ dx.$$

□

Fact 41
Let $A = \{(x,y) : x \in [a,b], y \text{ is between } f(x) \text{ and } g(x)\}$. Then the area of A is

$$\int_{x=a}^{b} |f(x) - g(x)|\ dx.$$

Reminder $|x|$ is x when $x \geq 0$ and $-x$ when $x \leq 0$.

Example

- Find the area between the curves $f(x) = x$ and $g(x) = x^2$ for x from 0 to 2.
- Note $f(x) - g(x) = x - x^2 = x(1-x)$. So for $x \in [0,1]$, $x \geq 0$ and $(1-x) \geq 0$ so $|f(x)-g(x)| = x(1-x)$.
- When $x \in [1,2]$, $x \geq 0$ but $(1-x) \leq 0$, so $|f(x) - g(x)| = -(f(x) - g(x)) = -x(1-x)$. Therefore

$$\text{area}(A) = \int_{x=0}^{1} x(1-x)\ dx + \int_{x=1}^{2} -x(1-x)\ dx$$
$$= 1$$

Proof. Let $A_1 = \{(x,y) : x \in [a,b] \text{ and } y \in [f(x), g(x)]\}$ and $A_2 = \{(x,y) : x \in [a,b] \text{ and } y \in [g(x), f(x)]\}$. Then

$$\int_{(x,y)\in A} 1\ d\mathbb{R}^2 = \mathbb{1}_{(x,y)\in A_1} 1\ d\mathbb{R}^2 + \mathbb{1}_{(x,y)\in A_2} 1\ d\mathbb{R}^2.$$

Note that

$$\begin{aligned}\int_{(x,y)\in A_1} 1 d\mathbb{R}^2 &= \int_{x\in\mathbb{R}} \mathbb{1}_{y\in\mathbb{R}}\mathbb{1}(x\in[a,b], y\in[f(x),g(x)])\ dydx \\ &= \int_{x\in\mathbb{R}} \mathbb{1}(x\in[a,b], f(x)\leq g(x))\mathbb{1}_{y=f(x)}^{g(x)}\ dy\ dx \\ &= \int_{x\in[a,b],f(x)\leq g(x)} g(x)-f(x)\ dx.\end{aligned}$$

Similarly,

$$\int_{(x,y)\in A_2} = \int_{x\in[a,b],g(x)\leq f(x)} f(x)-g(x)\ dx.$$

Putting these together gives

$$\int_{(x,y)\in A} = \int_{x=a}^{b} |g(x)-f(x)|\ dx.$$

□

26.2 Special cases of the volume integral

Integrate cross-sectional area to get volume...

Fact 42
Let $S \subset \mathbb{R}^3$ be compact with $\min\{z : (x,y,z) \in S\} = a$ and $\max\{z : (x,y,z) \in S\}$. Then

$$\text{volume}(S) = \int_{z=a}^{b} \text{area}(\{(x,y) : (x,y,z) \in S\})$$

[Note that the area of (x,y) such that $(x,y,z) \in S$ is the *cross-sectional area* of the solid.

Proof. The volume of S is

$$\begin{aligned}\int_{\mathbb{R}^3} \mathbb{1}((x,y,z)\in S)\ d\mathbb{R}^3 &= \int_{z\in\mathbb{R}}\int_{y\in\mathbb{R}}\int_{x\in\mathbb{R}} \mathbb{1}((x,y,z)\in S)\ dx\ dy\ dz \\ &= \int_{z\in[a,b]}\int_{(x,y):(x,y,z)\in S} 1\ d\mathbb{R}^2\ dz \\ &= \int_{z\in[a,b]} \text{area}(\{(x,y):(x,y,z)\in S\})\ dz.\end{aligned}$$

□

26.3 n-dimensional hypervolume

Fact 43
The set $A \subset \mathbb{R}^n$ has **Riemann length ($n = 1$), area ($n = 2$), volume ($n = 3$), or hypervolume** ($n > 3$) of

$$\int_A 1\ d\mathbb{R}^n = \int_{\mathbb{R}^n} \mathbb{1}(v \in A)\ d\mathbb{R}^n$$

when the Riemann integral exists.

27 Changing variables

Question of the Day Find the volume of the region above the cone $z \geq \sqrt{x^2+y^2}$ and inside the unit sphere $x^2+y^2+z^2 \leq 1$.

Today

- Transforming coordinate systems

Why different coordinate systems?

- Recall polar coordinate transformation:

$$x = r\cos(\theta),\ y = r\sin(\theta),\ r = \sqrt{x^2+y^2},\ \theta = \arctan(y/x).$$

- Differential polar coordinate transformation went along with it:

$$dx\ dy = r\ dr\ d\theta.$$

- Easier to write regions such as circles in polar coordinates:

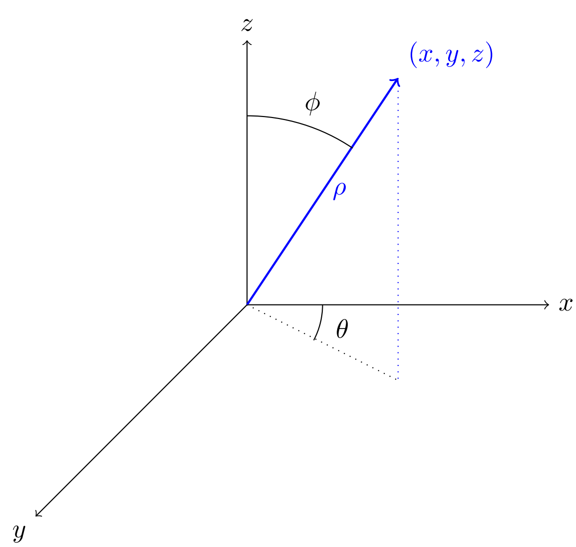
$$\{(x,y) : x^2+y^2 \leq 3\} = \{(r,\theta) : r \leq \sqrt{3}, \theta \in [0,2\pi]\}.$$

- In three dimensions, use spherical coordinates or cylindrical coordinates.

27.1 Spherical coordinates

- Represent a point (x, y, z) by

$$\begin{aligned} \rho &= \text{distance from origin} \\ \theta &= \text{angle in the } x\text{-}y \text{ plane} \\ \phi &= \text{angle from } z\text{-axis} \end{aligned}$$



- Dist. $(0,0,0)$ to $(0,0,z)$ is $\rho\cos(\phi)$
  Dist. from $(0,0,0)$ to $(x,y,0)$ is $\rho\sin(\phi)$.

Put these together to give

Definition 93
Spherical coordinates for a point in $\mathbb{R}^3$ is a 3-tuple (ρ, θ, ϕ), where

$$\begin{aligned} z &= \rho\cos(\phi) \\ x &= \rho\sin(\phi)\cos(\theta) \\ y &= \rho\sin(\phi)\sin(\theta) \end{aligned}$$

Just like with polar cooridnates, this has a differential transformation as well.

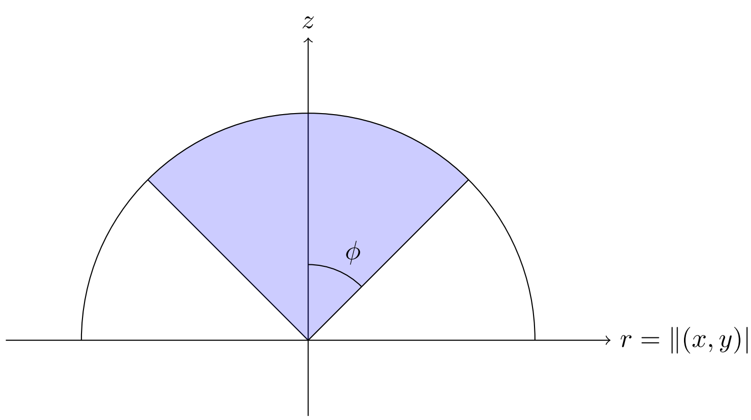
Fact 44
For the spherical coordinate transformation:

$$dx\ dy\ dz = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$$

[Intution: in 2-D $dx\ dy = r\ dr\ d\theta = \rho\sin(\phi)\ dr\ d\theta$. Extra factor of ρ comes from third dimension.]

Qotd

- Find volume in $(x, y, z) : z \geq \sqrt{x^2 + y^2}, x^2 + y^2 + z^2 \leq 1$.
- This region is symmetric around z-axis, so that gives $\theta \in [0, 2\pi]$. Side view:



- Using the spherical coordinate system $r = \rho\sin(\phi)$, $z = \rho\cos(\phi)$, so $z \geq r$ is $\sin(\phi) \geq \cos(\phi)$
- So the solid of interest is
$$\theta = [0, 2\pi],\ \phi = [0, \pi/4], \rho \in [0, 1].$$
The volume becomes:
$$\begin{aligned} \int_{(x,y,z)\in S} 1\ dx\ dy\ dz &= \int_{(\rho,\theta,\phi)\in[0,1]\times[0,\pi/4]\times[0,2\pi]} \rho^2\sin(\phi)\ d\phi\ d\theta\ d\rho \\ &= \int_{\rho\in[0,1]}\int_{\theta\in[0,2\pi]}\int_{\phi\in[0,\pi/4]} \rho^2\sin(\phi)\ d\phi\ d\theta\ d\rho \\ &= \int_{\rho\in[0,1]}\int_{\theta\in[0,2\pi]} \rho^2(-\cos(\phi))|_0^{\pi/4}\ d\theta\ d\rho \\ &= \int_{\rho\in[0,1]} (2\pi)\rho^2(1-\sqrt{2}/2)\ d\rho \\ &= (2\pi)(1-\sqrt{2}/2)\rho^3/3|_0^1 \\ &= \frac{2}{3}\pi\left(1-\frac{\sqrt{2}}{2}\right) \approx \boxed{0.6134}. \end{aligned}$$

27.2 Cylindrical coordinates

A different way of describing a point in $\mathbb{R}^3$ is called cylindrical coordinates, and you use them in a similar fashion to spherical coordinates. This method is just polar coordinates with the z-coordinate appended.

Definition 94
Cylindrical coordinates for a point in $\mathbb{R}^3$ are a 3-tuple (r, θ, z) where

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$z = z$$

As with spherical coordinates, there is a simple differential transformation:

Fact 45
For cylindrical coordinates

$$dx\ dy\ dz = r\ dr\ d\theta\ dz.$$

27.3 Linear approximations for $f : \mathbb{R}^m \to \mathbb{R}^n$

Recall

- For a real-valued function $f_1 : \mathbb{R}^m \to \mathbb{R}$, the best linear approximation is:

$$f(v + h) \approx f(v) + \nabla f \cdot h.$$

- So for $F : \mathbb{R}^m \to \mathbb{R}^n$, where

$$F(v) = \begin{pmatrix} f_1(v) \\ \vdots \\ f_n(v) \end{pmatrix}$$

the best linear approximation is

$$F(v + h) \approx F(v) + \begin{pmatrix} \nabla f_1(v) \cdot h \\ \nabla f_2(v) \cdot h \\ \vdots \\ \nabla f_n(v) \cdot h \end{pmatrix} = F(v) + DF(v)h.$$

Here DF is the $n \times m$ matrix whose ith row is ∇f_i. That makes the (i, j)th entry in the matrix $\partial f_i / \partial x_j$.

Definition 95
For $F : \mathbb{R}^m \to \mathbb{R}^n$ written as $F(v) = (f_1(v), \ldots, f_n(v))$, the **derivative** of F is the $n \times m$ matrix DF whose (i, j)th entry is $\partial f_i / \partial x_j$.

This derivative gives the best linear approximation of F in the following sense.

Fact 46
For $F : \mathbb{R}^m \to \mathbb{R}^n$ in C^1, there exists a vector field $\Phi_F : \mathbb{R}^m \to \mathbb{R}^n$ such that $\lim_{h \to 0 \in \mathbb{R}^m} \Phi_F(h) = (0, 0, \ldots, 0) \in \mathbb{R}^n$, and

$$F(v + h) = F(v) + DF(v)h + \|h\|\ \Phi_F(h).$$

In other words, $F(v) + DF(v)h$ is the best linear approximation to F near v.

Example

- Find the best linear approximation of $F(x,y) = (\sin(x), \sin(x+2y), xy)$ at $(0,0)$?
- $F(0,0) = (0,0,0)$, $v = (0,0)$, $h = (x,y)$, so now find the derivative matrix:

$$DF(0,0) = \begin{pmatrix} \partial(\sin(x))/\partial x & \partial(\sin(x))\partial y \\ \partial(\sin(x+2y))/\partial x & \partial(\sin(x+2y))\partial y \\ \partial(xy)/\partial x & \partial(xy)/\partial y \end{pmatrix}\Bigg|_{(0,0)}$$
$$= \begin{pmatrix} \cos(x) & 0 \\ \cos(x+2y) & 2\cos(x+2y) \\ y & x \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 0 \end{pmatrix}$$

So our linear approximation is

$$F_{\text{LA}}(x,y) = F(0,0) + \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \boxed{\begin{pmatrix} x \\ x+2y \\ 0 \end{pmatrix}}.$$

28 Determinants and Inverses

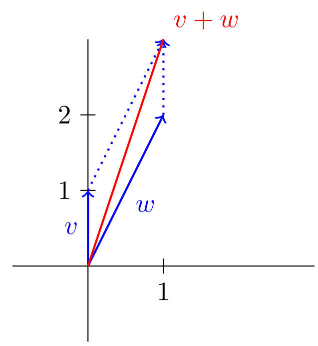
Question of the Day What is area spanned by the parallelogram with sides $v = (0,1)$, $w = (1,2)$?

Today

- Area spanned by vectors

Multiple ways to solve qotd

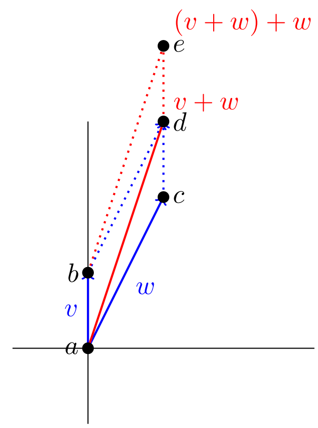
- First way: break parallelogram into two triangles:



- Each triangle has base of 1, height of 1, so total area is

$$(1/2)(1)(1) + (1/2)(1)(1) = 1.$$

- Let $\text{par}(v,w)$ be the parallelogram spanned by v and w. Then:
  Key observation: $\text{area}(\text{par}(v,w)) = \text{area}(\text{par}(v, v+w))$

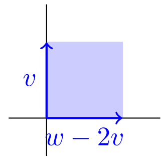


$$\begin{aligned}\text{area}\{v,w\} &= \text{area}(abd) + \text{area}(acd)\\ \text{area}\{v, v+w\} &= \text{area}(abd) + \text{area}(bde)\end{aligned}$$

But triangle acd is just the vector v plus the triangle acd! [So they must have the same area].

- Since $w = v + (w - v)$, this also gives:

$$\begin{aligned}\text{area}(\text{par}(v,w)) &= \text{area}(\text{par}(v, w-v))\\ &= \text{area}(\text{par}(v, w-2v)) = \text{area}(\text{par}((0,1),(1,0)) = 1\end{aligned}$$



- Note, $\text{area}(\text{par}(2.3v, w)) = 2.3\text{area}(\text{par}(v, w))$.

28.1 The determinant of a 2×2 matrix

Definition 96
For vectors (a, b) and (c, d), the **determinant** of the vectors is a function with three properties

1: $\det((0,1),(1,0)) = 1$

2: $(\forall \alpha, \beta \in \mathbb{R})(\forall v, w \in \mathbb{R}^2)(\det(\alpha v, \beta w) = \alpha\beta \det(v, w))$

3: $(\forall v, w \in \mathbb{R}^2)(\det(v, w) = \det(v + w, v) = \det(v, v + w))$

When the vectors are the rows of a square matrix A, then the determinant of the vectors is also called the determinant of A.

Fact 47
The area of the parallelogram spanned by v and w is $|\det(v, w)|$.

Fact 48
For a 2×2 matrix,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

[The proof requires that we show that this function has the three properties, and that any function with these three properties must have this exact form, and you will see it in Linear Algebra.]

Example

- Find the area of the parallelogram spanned by $(2, -3)$ and $(6, -5)$:

$$\left|\det \begin{pmatrix} 2 & -3 \\ 6 & -5 \end{pmatrix}\right| = |(2)(-5) - (-3)(6)| = |3| = \boxed{3}.$$

28.2 Inverses of matrices

- Determinants are useful because they

  1: Allow us to determine when $Av = b$ has a solution for b

  2: Allow use to do general change of variables in integrals

- Today we'll concentrate on the $Av = b$ problem.
- Recall that $v \mapsto Av$ is a linear transformation:

$$(\forall a, b \in \mathbb{R})(\forall v, w \in \mathbb{R}^n)(A(av + bw) = aAv + bAw).$$

When $Av = b$ always has a solution, that means that the function that maps v to Av is *invertible*, and the function that maps Av back to v is called the inverse function.

Fact 49
If $v \mapsto Av$ has an inverse function, that inverse is also invertible, so can be written $A^{-1}w$ for some matrix A^{-1}.

[Proof in Linear Algebra.]

- From the definition of an inverse function, $A^{-1}(Av) = v$. From the definition of matrix multiplication, $A^{-1}(Av) = (A^{-1}A)v$, so $A^{-1}A$ must equal the indentity matrix I. The (i,j)th entry of I is 1 if $i = j$, and 0 otherwise.

Definition 97
For A an $n \times n$ matrix, if $A^{-1}A = I$, say that A^{-1} is the **inverse** of the matrix A.

Fact 50
If A has an inverse A^{-1}, then the solution to the system of equations $Av = b$ is $v = A^{-1}b$.

- Not all matrices have inverses. If
$$Av = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
there is no solution for v, and so there cannot exist a matrix A^{-1}.

- The determinant characterizes if a matrix has a solution.

Fact 51
A matrix A has an inverse if and only if $\det(A) \neq 0$.

Examples
$$\det \begin{pmatrix} 3 & 4 \\ -6 & -8 \end{pmatrix} = 0, \text{ no inverse }, \det \begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} = 3, \text{ has inverse.}$$

- What if the function is not linear?

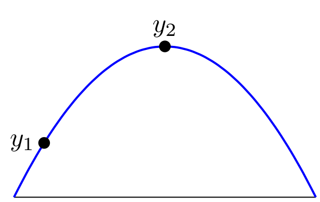
- Then can only say that locally it does not have an inverse, by using the linear approximation as a proxy.

Theorem 10 (Inverse function theorem)
Let $F : U \to \mathbb{R}^n$ where $U \subseteq \mathbb{R}^n$ is open be in C^1. if DF has an inverse at v, there exists $\epsilon > 0$ such that for all z within distance ϵ of v, there exists a unique solution w such that $F(w) = z$. (Write $w = F^{-1}(z)$.)

- Say that F is *locally invertible* near $F(w)$.

- Very useful when it comes to showing the systems of differential equations has a solution.

- In one dimension, easy to picture with function $f(x) = 1 - x^2$. Note that $f'(x) \neq 0$ (so invertible) whenever $x \neq 0$. So function always has unique solution as long as $f(x) \neq 1$. However, for $f(x) = 1$, $x = 0$, and all nearby y-coordinates have 2 solutions!



Change y_1 a little bit, know how to change $x : f(x) = y_1$. But change y_2 a little bit, two solutions to $x : f(x) = y_2$.

29 Change of Variables Formula

Question of the Day Find

$$\int_D \left(\frac{x-y}{x+y}\right)^4 \; dy \; dx$$

where D is the triangular region between $(0,0)$, $(1,0)$, and $(0,1)$.

Today

- Custom design of coordinate systems

Making a problem simpler

- Change variables to another coordinate system
- LID

  1: Limits

  2: Integrand

  3: Differential

- Today: how to change the differential

1-D

- Recall change of variables: for $w = f(x)$, $dw = f'(x)\; dx$.
- Example: solve $\int_{x\in[0,5]} 2x\exp(-x^2)\; dx$.
- Let $w = x^2$ to make exponential easier.
- Then $x \in [0,5]$ means $w \in [0,25]$.
- Also $dw = 2x\; dx$. So

$$\begin{aligned}
\int_{x\in[0,5]} 2x\exp(-x^2)\; dx &= \int_{w\in[0,25]} 2x\exp(-x^2)\; dx && \text{[Change limits]}\\
&= \int_{w\in[0,25]} \exp(-x^2)\; dw && \text{[Change differential]}\\
&= \int_{w\in[0,25]} \exp(-w)\; dw && \text{[Change integrand]}\\
&= \exp(-w)/(-1)|_0^{25}\\
&= 1-\exp(-25) \approx 1.000
\end{aligned}$$

29.1 Change of variables in n dimensions

Theorem 11 (Change of variables theorem)
Suppose $F(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$. Then

$$dy_1\; dy_2\; \cdots\; dy_n = |\det DF(x_1,\ldots,x_n)| dx_1\; dx_2 \cdots\; dx_n$$

Definition 98
The **Jacobian** of a function F is $|\det DF|$.

Why?

- In 2-D, remember $|\det DF|$ is area spanned by vectors made from rows of DF. So $|\det DF|\ dx_1\ dx_2$ is area spanned by vectors where one vectors is length dx_1 and other is length dx_2.
- In n dimensionsl, $|\det(A)|$ is hyper volume of region spanned by the rows of A. Then multiply by all the dx_i to scale the vectors

Example: qotd

- Use transformation to make the integrand simpler:

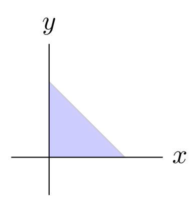
$$s = x - y,\ t = x + y$$

Then $DF = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, so $|\det DF| = |(1)(1) - (-1)(1)| = 2$, so

$$ds\ dt = 2\ dx\ dy.$$

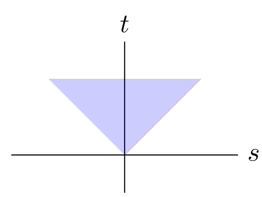
- Now limits. In terms of x and y:

$$D = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}.$$



Note that $x = (1/2)(s + t)$, $y = (1/2)(t - s)$. So

$$D = \{(s, t) : (1/2)(s + t) \geq 0, (1/2)(t - s) \leq 0, t \leq 1\}$$



- Transformed integral:

$$\begin{aligned} I &= \int_{(s,t):s+t\geq 0, s-t\geq 0, t\leq 1} (s/t)^4\ (1/2)\ ds\ dt \\ &= (1/2) \int_{t\in[0,1]} \int_{s\in[-t,t]} (s/t)^4\ ds\ dt \\ &= (1/2) \int_{t\in[0,1]} s^5 t^{-4} (1/5)|_{-t}^{t}\ dt \\ &= (1/2) \int_{t\in[0,1]} 2t/5\ dt \\ &= (1/2)(t^2/5)|_0^1 \\ &= 1/10 = \boxed{0.1000} \end{aligned}$$

Example changing differentials

- For $s = u^2v^2$ and $t = v^2 - u^2$, find $ds\ dt$ in terms of $dv\ du$.
- Create the determinant:
$$DF = \begin{pmatrix} \partial s/\partial u & \partial s/\partial v \\ \partial t/\partial u & \partial t/\partial v \end{pmatrix} = \begin{pmatrix} 2uv^2 & 2u^2v \\ -2u & 2v \end{pmatrix}$$
so
$$|\det DF| = |(2uv^2)(2v) - (2u^2v)(-2u)| = |4uv^3 + 4u^3v|.$$
Note that the absolute value depends on the values of u and v:
$$\begin{aligned} |4uv^3 + 4u^3v| = (4uv^3 + 4u^3v)\mathbb{1}(u \geq 0, v \geq 0) + \\ (4uv^3 + 4u^3v)\mathbb{1}(u < 0, v < 0) - \\ (4uv^3 + 4u^3v)\mathbb{1}(u < 0, v \geq 0) - \\ (4uv^3 + 4u^3v)\mathbb{1}(u \geq 0, v < 0). \end{aligned}$$

29.2 Polar coordinates transformation

To practice using the Jacobian, let's use it to derive the formula:
$$dx\ dy = r\ dr\ d\theta.$$

- Here x and y are functions of r and θ:
$$\begin{aligned} x &= r\cos(\theta) = f_1(r, \theta) \\ y &= r\sin(\theta) = f_2(r, \theta), \end{aligned}$$
so for $F(r, \theta) = (f_1(r, \theta), f_2(r, \theta))$,
$$DF = \begin{pmatrix} \partial f_1/\partial r & \partial f_1/\partial \theta \\ \partial f_2/\partial r & \partial f_2/\partial \theta \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}$$
So
$$|\det DF| = |r\cos^2(\theta) - (-r)\sin^2(\theta)| = |r(\cos^2(\theta) + \sin^2(\theta)| = r.$$

29.3 n dimensional transformations

- This works for $n \geq 3$ dimensional transformations as well.
- Need to be able to calculate the determinant of an n by n matrix is.
- Just apply the rules in Definition 96 to get answer.

Fact 52
For a 3 by 3 matrix, the determinant is
$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - afh - bdi - ceg.$$

Notes

- In general, the determinant of an n by n matrix has $n!$ terms in the formula, so gets large very quick. In linear algebra, you will learn a more efficient ways to compute the determinant of a matrix such as Gaussian Elimination and QR factorization.
- Can use this to show that for the spherical coordinate transformation, $|\det DF| = \rho^2 \sin(\phi)$.

30 Implicit Functions

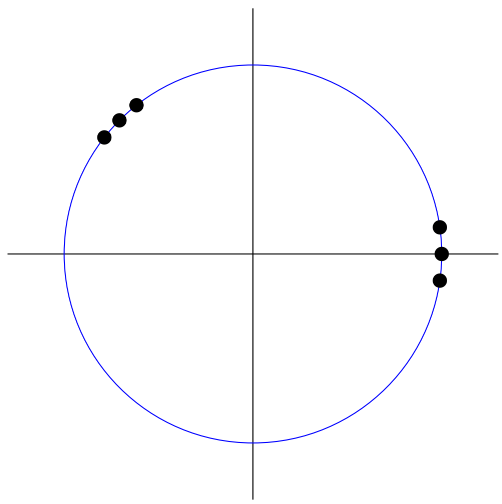
Question of the Day Consider the set of points (x, y) such that $x^2y + 3y^3x^4 - 4 = 0$. For instance $(1, 1)$ is a point that satisfies this. For x values near 1, is there a unique solution for y that makes (x, y) near $(1, 1)$?

Today

- The implicit function theorem

Example

- Consider the points (x, y) that satisfy $x^2 + y^2 = 1$.
- This is an implicitly defined curve, and the point $(1, 0)$ is on the curve.
- For x near 1, there are two solutions for y near $(1, 0)$
- Now consider the point $(-1/\sqrt{2}, 1/\sqrt{2})$. For x near $-1/\sqrt{2}$, there is only a single solution for y.



Qotd

- We know that $(1, 1)$ solves $x^2y + 3y^3x^4 - 4 = 0$. What happens for x slightly larger than 1? Slightly smaller? Is the y unique?

Definition 99
Consider the set of points that satisfies $f(x, y) = c$. If there is a unique solution y given x, call $y = \phi(x)$ the function **implicity determined** by f.

- Remember circle example: $f(x, y) = x^2 + y^2$. Note $\partial f/\partial y = 2y$. At $y = 0$, $\partial f/\partial y = 0$. Turns out that's the only situation where there might not be an implicit function.

30.1 The Implicit function theorem

For the implicit equation $f(x_1, \ldots, x_n) = c$, if I pick a variable x_i and ask if there is a unique solution for x_i in terms of the other variables near some point $a = (a_1, \ldots, a_n)$, then

Theorem 12 (Implicit Function Theorem)
Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be in C^1. Let $a = (a_1, \ldots, a_n) \in U$, where $f(a) = c$, and $D_i f(a) = \partial f / \partial x_i|_a \neq 0$. Then there exists δ such that for all points within distance δ of $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, there is a unique solution for x_i satisfying $f(x_1, \ldots, x_n) = c$ that can be called

$$x_i = \phi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Moreover, $\phi \in C^1$.

Notes

- If $\partial f/\partial y \neq 0$, then ϕ always exists.
- If $\partial f/\partial y = 0$, then ϕ might or might not exist.
  1: Example. $f(x, y) = x^2 + y^2 = 1$ at $(1, 1)$, $\partial f/\partial y = 0$, and no inverse.
  2: Example. $f(x, y) = y^3 - x = 0$ at $(0, 0)$, $\partial f/\partial y = 3y^2 = 0$, but $y = \phi(x) = x^{1/3}$ has a unique inverse everywhere!

Qotd

- Set of points: $f(x, y) = x^2 y + 3y^3 x^4 - 4 = 0$ near $(1, 1)$.
- $\partial f/\partial y = x^2 + 9y^2 x^4$. Plug in $(1, 1)$ to get $(\partial f/\partial y)(1, 1) = 10 \neq 0$.
- Hence $y = \phi(x)$ exists!
- Note that $\phi(x)$ might be hard to find though...
- On the other hand, finding $\phi'(x)$ in terms of x and y is easy!
  1: Start with $y = \phi(x)$.
  2: Differentiate both sides of $f(x, y) = f(x, \phi(x)) = c$ using the chain rule.
  3: Substitute back in $y = \phi(x)$.

$$\begin{aligned}
0 &= \frac{d}{dx} f(x, \phi(x)) \\
&= \frac{d}{dx}[x^2\phi(x) + 3\phi(x)^3 x^4 - 4 \\
&= 2x\phi(x) + x^2\phi'(x) + 9\phi(x)^2\phi'(x)x^4 + 12\phi(x)^3 x^3 \\
0 &= 2xy + x^2\phi'(x) + 9y^2\phi'(x)x^4 + 12y^3x^3 \\
-2xy - 12y^3x^3 &= \phi'(x)[x^2 + 9y^2x^4] \\
\phi'(x) &= \frac{-2xy - 12y^3x^3}{x^2 + 9y^2x^4}.
\end{aligned}$$

- Applications similar to Inverse function theorem.
- Often used to show that differential equations have unique solutions.
- Works equally well for x as for y: If $(\partial f/\partial x)(v) \neq 0$, then can solve $f(x, y) = c$ for x near v.

30.2 Proof of the Implicit Function Theorem

Proof. Given $f(x, y) = c$ is our set of points, let $F(x, y) = (x, f(x, y))$ be from $\mathbb{R}^2$ to $\mathbb{R}^2$. Then

$$DF(x, y) = \begin{pmatrix} 1 & 0 \\ \partial f/\partial x & \partial f/\partial y \end{pmatrix}.$$

So $\det DF = 0 \Leftrightarrow \partial f/\partial y = 0$.

By the Inverse function theorem, that tells us that $\partial f/\partial y = 0$ means that $F(x, y)$ is locally invertible. Consider the point (a, b) where $f(a, b) = c$. Then $F(a, b) = (a, f(a, b)) = (a, c)$.

Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be the inverse of $F : \mathbb{R}^2 \to \mathbb{R}^2$. Then

$$G(F(x, y)) = (x, y),$$

since it is an inverse. But $G(F(x, y)) = G(x, f(x, y)) = (x, y)$. So the first output of G is the same as the first input.

That means we can write

$$G(s, t) = (s, g(s, t)),$$

where $g(s, t) : \mathbb{R}^2 \to \mathbb{R}$.

Now let $\phi(x) = g(x, c)$. Then we have:

$$\begin{aligned} F(x, \phi(x)) &= F(x, g(x, c)) \\ &= F(G(x, c)) \\ &= (x, c). \end{aligned}$$

By the definition of F, $F(x, \phi(x)) = (x, f(x, \phi(x)))$. The only way these both can be true is if $c = f(x, \phi(x))$, which makes $\phi(x)$ the inverse function that we are looking for! $\square$

31 The Chain Rule

Question of the Day Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^r$. What should $DF_{G\circ F}$ look like?

Today

- The chain rule for vector fields

Chain rule in 1-D

- Recall linear approximations:
$$f(x+h) = f(x) + hf'(x) + o(h)$$
$$g(w+j) = g(w) + jg'(w) + o(j).$$
So what is $g(f(x+h))$?
$$g(f(x+h)) = g(f(x) + hf'(x) + o(h)).$$
Set $w = f(x)$ and $j = hf'(x) + o(h)$. Then
$$\begin{aligned} g(f(x+h)) &= g(w+j) \\ &= g(w) + jg'(w) + o(j) \\ &= g(f(x)) + (hf'(x) + o(h))g'(w) + o(hf'(x) + o(h)) \\ &= g(f(x)) + hf'(x)g'(w) + o(h)[g'(f(x)) + f'(x)] \\ &= g(f(x)) + hf'(x)g'(f(x)) + o(h), \end{aligned}$$
where we can ignore the $g'(f(x) + f'(x))$ because it is just a constant.
- That means the derivative of $g(f(x)) = [g \circ f](x)$ must be $g'(f(x))f'(x)$, or
$$[g \circ f]' = [g' \circ f]f'.$$

31.1 The chain rule for vector fields

Fact 53 (Chain rule for vector fields)
Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^r$ be in C^1. Then
$$[G \circ F]' = [DG \circ F]DF.$$

- This is one of those nice theorems where n-D form and 1-D form are exactly the same!
- Generalizes the chain rule that we had earlier for curves, $P : \mathbb{R} \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}$,
$$[g \circ C]' = (\nabla g \circ C) \cdot C'.$$
- When $g : \mathbb{R}^n \to \mathbb{R}$, $Dg = \nabla g$. And matrix multiplication when you have a row vector times a column vector is the same as the dot product.

Chain rule for vector fields. Use the linear approximation idea:
$$\begin{aligned} G(F(v+h)) &= G(F(v) + DF(v)h + \|h\|\,\Phi_F(v)) \\ &= G(F(v)) + DG(F(v))[DF(v)h + \|h\|\,\Phi_F(v)] + \\ &\quad \|DF(v)h + \|h\|\,\Phi_F(v)\|\,\Phi_G(DF(v)h + \|h\|\,\Phi_F(v)). \end{aligned}$$
A fact from linear algebra gives that $\|DF(v)h\| \leq M\,\|h\|$ for some constant M. Because Φ_G and Φ_F both go to 0 as h goes to zero, using this fact gives us that the last term can be written as $\|h\|\,\Phi(h)$, where $\Phi(h)$ is a matrix that is going to 0 as h goes to 0. Hence
$$\begin{aligned} G(F(v+h)) &= G(F(v) + DF(v)h + \|h\|\,\Phi_F(v)) \\ &= G(F(v)) + DG(F(v))DF(v)h + \|h\|\,[\Phi_F(v) + \Phi(h)], \end{aligned}$$
which means that $DG(F(v))DF(v)$ must be the derivative of $G(F(v))$. □

31.2 Hessian as second derivative

- Recall earlier we treated the Hessian of a real valued function as a second derivative in finding out if a critical point was a local max or local min.
- $g : \mathbb{R}^n \to \mathbb{R}$:

$$\begin{aligned} \nabla g = 0,\ Hg \text{ positive definite } &\Rightarrow \text{local min} \\ \nabla g = 0,\ Hg \text{ negitive definite } &\Rightarrow \text{local max.} \end{aligned}$$

- Today we note: $Hg = D(\nabla g)$. It is a second derivative!
- First derivative of g is ∇g, a row vector.
- Second derivative of g is $n \times n$ matrix.
- Third derivative of g is a three dimensional array.
  [Sometimes called a tensor, since tensors can also be represented by multidimensional array.]
- Second derivative: Hg has (i,j)th entry $D_j D_i g$.
- Third derivative: (i,j,k)th entry $D_k D_j D_i g$.
- Can make fourth and higher derivatives as well!
- We'll stick to second derivatives in this class.

31.3 Example

Suppose $F(x,y) = (x^2 - y, x + y)$ and $G(r,s) = (r - s, r + s)$.

1: What is $G(F(x,y))$?

2: What is $D[G \circ F]$?

To answer the first question, we treat the outputs of F as the inputs of G. So

$$G(F(x,y)) = G(x^2 - y, x + y) = (x^2 - x - 2y, x^2 + x).$$

To answer the section question, we could directly find the derivative of $G(F(x,y))$:

$$D[G \circ F](x,y) = \begin{matrix} \\ x^2-x-2y \\ x^2+x \end{matrix} \begin{matrix} \begin{matrix} x & \quad y \end{matrix} \\ \begin{pmatrix} 2x-1 & -2 \\ 2x+1 & 0 \end{pmatrix} \end{matrix}$$

Or we could use the chain rule!

$$DF = \begin{matrix} \\ x^2 - y \\ x + y \end{matrix} \begin{matrix} \begin{matrix} x & \quad y \end{matrix} \\ \begin{pmatrix} 2x & -1 \\ 1 & 1 \end{pmatrix} \end{matrix}, \quad DG = \begin{matrix} \\ r - s \\ r + s \end{matrix} \begin{matrix} \begin{matrix} r & \quad s \end{matrix} \\ \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \end{matrix}.$$

Since DG is constant, $[DG \circ F](x,y) = DG = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

So the chain rule gives:

$$\begin{aligned} D[G \circ F](x,y) &= [DG \circ F](x,y) DF(x,y) \\ &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2x & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2x-1 & -2 \\ 2x+1 & 0 \end{pmatrix}. \end{aligned}$$

Problems

31.1: Suppose $H(r, \theta) = (r\cos(\theta), r\sin(\theta))$ and $G(x, y) = (xy, x^2 + y^2)$.

(a) What is
$$[G \circ H](r, \theta)?$$

(b) Find $D[G \circ H]$ directly.

(c) Find $D[G \circ H]$ using the chain rule.

32 Parameterizing Surfaces

Question of the Day Parameterize the surface of a sphere of radius ρ.

Today

- Describing functions: Implicit v. parameterizations

Curves

- Implicit: $f(x, y) = c$, for example

$$x^2 + y^2/4 = 2, \; x - y^2 = 6$$

- Parameterized curve: $C(t) = (x(t), y(t))$. [Call t the parameter.] For example

$$(\sin(t), \cos(t))$$

Implicit description

- Implicit: $f(x, y, z) = c$, for example

$$x + 2y^2 - z = 4$$

- Parameterized surface: $f(t, r) = (x(t, r), y(t, r), z(t, r))$. [Call t and r the parameters.] For example

$$Y(t) = (t, t^2, e^t)$$

When does an implicit description equal a parameterization?

- When for each (x, y, z) satisfying $f(x, y, z) = c$, there exists t such that $x = x(t)$, $y = y(t)$, and $z = z(t)$. For example, if

$$A = \{(x, y) : x^2 + y^2 = 1\}, \; B = \{(\cos(t), \sin(t)) : t \in [0, 2\pi]\},$$

then $A = B$.

- Why? Well $B \subseteq A$ since $\sin^2(t) + \cos^2(t) = 1$ For x and y such that $x^2 + y^2 = 1$, then setting $t = \arctan(y/x)$ if $x \geq 0$ and $t = \arctan(y/x) + \pi$ if $x < 0$ gives $(\cos(t), \sin(t)) = (x, y)$. So $A \subseteq B$.

32.1 Parameterizing a sphere

Describing a sphere

- Start with implicit description
  - Let ρ be the radius of the sphere. Then the implicit description is: $x^2 + y^2 + z^2 = \rho^2$
- Use spherical coordinates to parameterize the surface.
  - Spherical coordinates use parameters θ, ϕ, ρ:

$$\begin{aligned} x &= \rho \sin(\phi) \cos(\theta) \\ y &= \rho \sin(\phi) \sin(\theta) \\ z &= \rho \cos(\phi). \end{aligned}$$

  Here ρ is fixed, so the parameters are ϕ and θ.

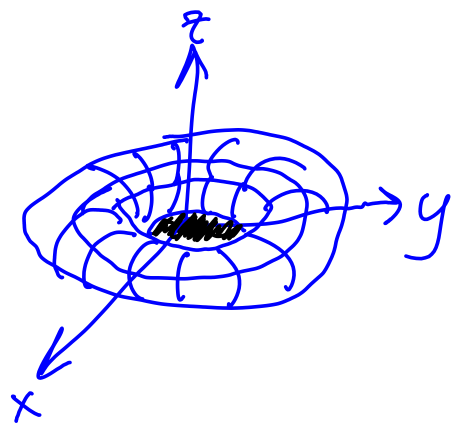
$$S(\phi, \theta) = \rho(\sin(\theta) \cos(\theta), \sin(\theta) \cos(\theta), \cos(\phi)).$$

- To parameterize a curve, need single parameter (like t)
- To parameterize a surface, need two parameters (like (ϕ, θ))

32.2 Torus, cone, paraboloid, cylinder, ellipsoid

Examples

- A torus (doughnut)



Parameterize

$$x = (a + b\cos(\phi))\cos(\theta)$$
$$y = (a + b\cos(\phi))\sin(\theta)$$
$$z = b\sin(\phi)$$

Constants

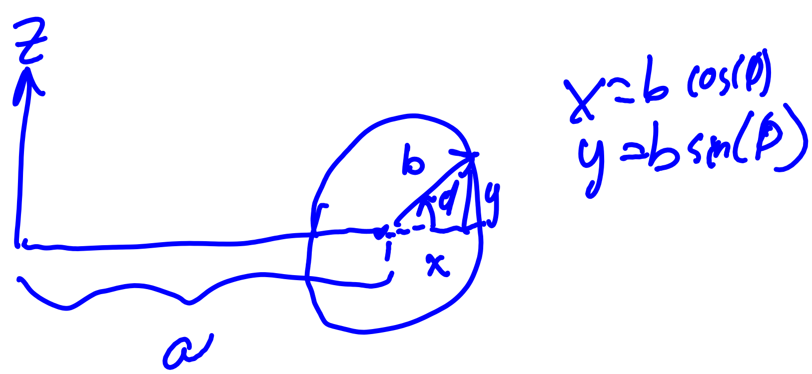
$$a = \text{distance from origin to center of cross section}$$
$$b = \text{radius of cross section}$$

- How the parameters work:

$$\theta \in [0, 2\pi] = \text{Rotates cross section around } z\text{-axis}$$
$$\phi \in [0, 2\pi) = \text{Makes cross section a circle}$$

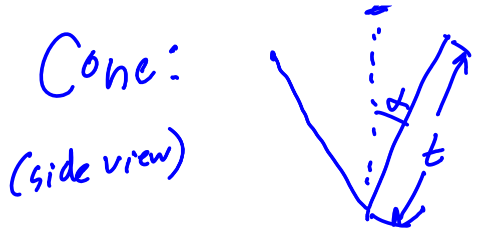
Note that ϕ here is different from ϕ in spherical coordinates!

- Cross section:



Cone

- Constant angle with z-axis, call it α. Parameters are t which is distance from the origin, and θ equals angle of rotation around z-axis.



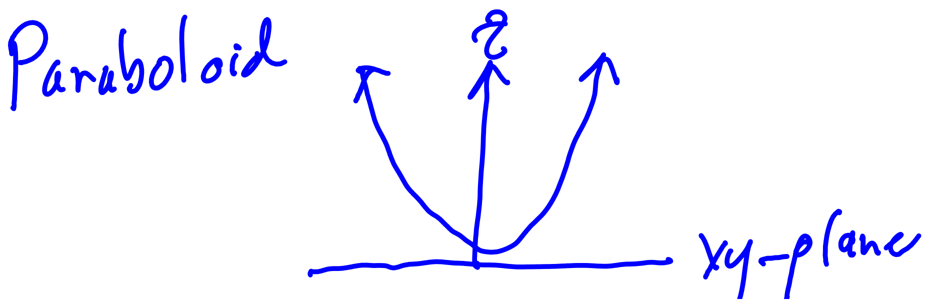
Parameterize

$$z = t\cos(\alpha)$$
$$x = t\sin(\alpha)\cos(\theta)$$
$$y = t\sin(\alpha)\sin(\theta)$$

Written another way:

$$S(t,\theta) = (t\sin(\alpha)\cos(\theta), t\sin(\alpha)\sin(\theta), t\cos(\alpha)).$$

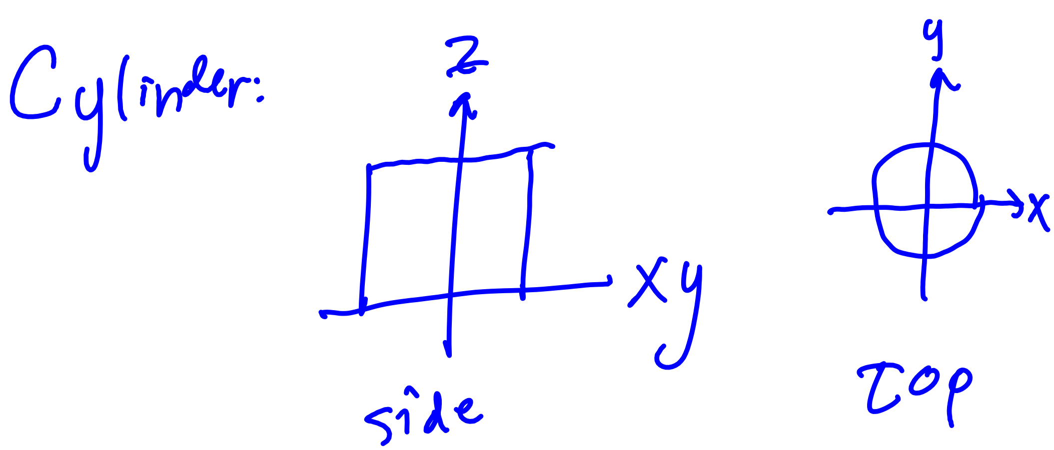
Paraboloid



For α a constant, and parameters $t \geq 0$, $\theta \in [0, 2\pi)$, let

$$S(t,\theta) = (\alpha t\cos(\theta), \alpha t\sin(\theta), t^2).$$

Cylinder



For a a constant, and parameters $\theta \in [0, 2\pi)$, $z \in \mathbb{R}$, let

$$S(\theta, z) = (a\cos(\theta), a\sin(\theta), z)$$

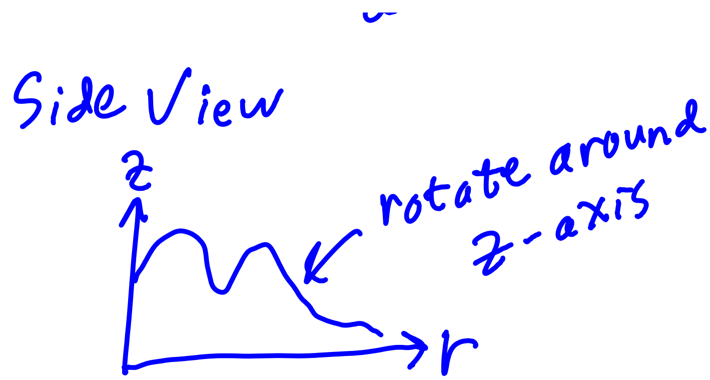
Ellipsoid

- Like a stretched sphere.
- Need three constants for scaling in each of three dimensions, a, b, c, two parameters $\phi \in [0, \pi]$, $\theta \in [0, 2\pi)$.

$$S(\phi, \theta) = (a \sin(\phi) \cos(\theta), b \sin(\phi) \sin(\theta), c \cos(\phi)).$$

General surface of revolution

- Want to rotate $f(r)$ around the z-axis:



$$S(r, \theta) = (r \cos(\theta), r \sin(\theta), f(r))$$

33 Finding the area of a surface

Question of the Day Find the surface area of a paraboloid $z = x^2 + y^2$ for $z \in [0, 2]$.

Today

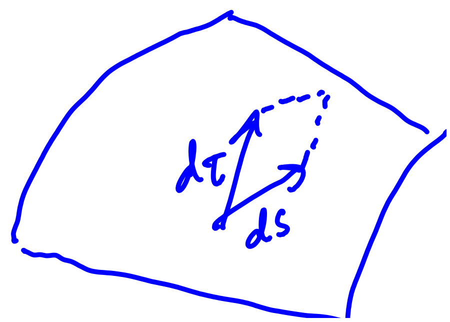
- Finding areas of surfaces

Setting up the integral

- First step: parameterize the surface. Ex, surface of a sphere:

$$S(t, s) = (\sin(t)\cos(s), \sin(t)\sin(s), \cos(t)).$$

Next, let $||dS||$ be the area of the parallelogram spanned by differential elements, dt and $d\theta$.



- Add up the area (integrate) to get total area:

$$S = \int_{t \in [0,\pi]} \int_{\theta \in [0,2\pi]} ||dS||$$

- The question: what is relationship between dS and $dt\ ds$?
- Recall, for ds the length of a differential element of a curve $C(t) \in C^1$,

$$ds = \|C'(t)\|\ dt.$$

Want a similar formula for dS.

33.1 Finding the area spanning by two vectors

- In 2-D, used the determinant to get this area.
- In 3-D, determinant gives volume of parallelepiped.
- Let $v, w \in \mathbb{R}^3$. If a is perpendicular to both v and w, and $\|a\| = 1$, then

$$\det \begin{pmatrix} a \\ v \\ w \end{pmatrix}$$

is the area spanned by v and w.

33.2 Cross products

Definition 100
For $v = (v_1, v_2, v_3)$, and $w = (w_1, w_2, w_3)$, let $v \times w$ be the **cross product** of v and w, where the formula is

$$v \times w = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1).$$

Note: v, w, and $v \times w$ are all elements of $\mathbb{R}^3$.

Fact 54
$v \times w$ is perpendicular to both v and w.

Proof. Note

$$\begin{aligned}(v \times w) \cdot v &= (v_2w_3 - v_3w_2)v_1 + (v_3w_1 - v_1w_3)v_2 + (v_1w_2 - v_2w_1)v_3 \\ &= v_2w_3v_1 - v_3w_2v_1 + v_3w_1v_2 - v_1w_3v_2 + v_1w_2v_3 - v_2w_1v_3 \\ &= 0.\end{aligned}$$

So $v \times w$ and v are perpendicular. A similar calculation shows that $v \times w$ and w are perpendicular. □

Mnemonic for cross product

- Note $v_2w_3 - v_3w_2$ is derivative of $\begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix}$.

Fact 55
The cross product of $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ can be written as

$$\det \begin{pmatrix} (1,0,0) & (0,1,0) & (0,0,1) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

Of course, using this mnemonic requries knowing how to compute determinants of 3 by 3 matrices!

Fact 56
The area of the parallelogram spanned by v and w in $\mathbb{R}^3$ is $\|v \times w\|$.

Proof. Let a be the area of the parallelogram spanned by v and w. Then since $v \times w$ is perpendicular to v and w, $a \|v \times w\|$ is the volume of the parallelepiped spanned by v, w, and $v \times w$.

This can also be found using the determinant, which gives

$$\begin{aligned}a \|v \times w\| &= \left| \det \begin{pmatrix} v_2w_3 - v_3w_2 & v_3w_1 - v_1w_3 & v_1w_2 - v_2w_1 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \right| \\ &= [v_2^2w_3^2 - v_2v_3w_2w_3] + [v_3^2w_1^2 - v_1v_3w_1w_3] \\ &\quad + [v_1^2w_2^2 - v_1v_2w_1w_2] - [v_2v_3w_2w_3 - v_3^2w_2^2] \\ &\quad - [v_1v_3w_1w_3 - v_1^2w_3^2] - [v_1v_2w_1w_2 - v_2^2w_1^2] \\ &= v_2^2w_3^2 + v_3^2w_1^2 + v_1^2w_2^2 + v_3^2w_2^2 + v_1^2w_3^2 + v_2^2w_1^2 \\ &\quad - 2v_1v_2w_1w_2 - 2v_1v_3w_1w_3 - 2v_2v_3w_2w_3.\end{aligned}$$

This last expression is the same as

$$\|v \times w\|^2 = (v_2w_3 - v_3w_2)^2 + (v_3w_1 - v_1w_3)^2 + (v_1w_2 - v_2w_1)^2$$

Hence $a \|v \times w\| = \|v \times w\|^2$ which means either $v \times w = 0$ or $a = \|v \times w\|$. It is straightforward to check that if $\|v \times w\| = 0$, then $w = \lambda v$, so $a = 0$ as well. □

33.3 Surface integrals

This gives the following fact:

Fact 57
Let $S(t, s)$ be a parameterized surface. Then

$$||dS|| = \left\| \frac{\partial S}{\partial t} \times \frac{\partial S}{\partial s} \right\| \; ds \; dt.$$

Qotd

- First parameterize the paraboloid (multiple ways to do this, this way makes later calculations easier)

$$S(x, y) = (x, y, x^2 + y^2),\ \{(x, y) : x^2 + y^2 \leq 2\}.$$

- Next find $\partial S/\partial x$ and $\partial S/\partial y$:

$$\frac{\partial S}{\partial x} = (1, 0, 2x),\ \frac{\partial S}{\partial y} = (0, 1, 2y).$$

- Find $\partial S/\partial x \times \partial S/\partial y$:

$$\begin{aligned}\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} &= ((0)(2y) - (2x)(1), (2x)(0) - (1)(2y), (1)(1) - (0)(0)) \\ &= (-2x, -2y, 1).\end{aligned}$$

- Next, find $\|\partial S/\partial x \times \partial S/\partial y\|$:

$$\left\| \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right\| = [(-2x)^2 + (-2y)^2 + 1^1]^{1/2} = (4(x^2 + y^2) + 1)^{1/2}.$$

- Now we have an integral!

$$S = \int_{z \in [0,2]} dS = \int_{(x,y):x^2+y^2 \leq 2} (4(x^2 + y^2) + 1)^{1/2} \; dx \; dy.$$

- Best way to solve this integral: convert to polar coordinates.

$$r^2 = x^2 + y^2,\ dx \; dy = r \; dr \; d\theta.$$

  So

$$\begin{aligned}S &= \int_{r \leq 2} \int_{\theta \in [0,2\pi]} (4r^2 + 1)^{1/2} r \; d\theta \; dr \\ &= 2\pi \int_{r \leq 2} (4r^2 + 1)^{1/2} r \; dr \\ &= 2\pi \left. \frac{(4r^2 + 1)^{3/2}}{(3/2)8} \right|_{r=0}^{r=2} \\ &= 2\pi[9^{3/2} - 1^{3/2}]/12 = (13/3)\pi \approx \boxed{13.61}.\end{aligned}$$

33.4 The curl of a vector field

The cross product also shows up in the *curl* of a vector field $F(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$. The curl is the three dimensional equivalent of the rotation.

Definition 101
For $F : \mathbb{R}^3 \to \mathbb{R}^3$, the **curl** of F is

$$\begin{aligned}(\nabla \times F) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (f_1, f_2, f_3) \\ &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right).\end{aligned}$$

34 More Surface Integrals

Question of the Day Find the surface area of the cone $z = r$ for $z \in [0, 1]$.

Today

- More examples of surface integrals.

Finding surface integrals

- To find $\int_{(t,s)\in A} ||dS||$:
  - **1:** Parameterize the surface at $S(t, s)$
  - **2:** Find $\|\partial S/\partial t \times \partial S/\partial s\|$
  - **3:** Solve the integral

$$\int_{(t,s)\in A} ||dS|| = \int_{(t,s)\in A} \left\| \frac{\partial S}{\partial t} \times \frac{\partial S}{\partial s} \right\| \, ds \, dt$$

Qotd

- Always several ways to parameterize surface
- Using x and y:

$$S_1(x, y) = (x, y, \sqrt{x^2 + y^2})$$

- Using polar coordinates

$$S_2(r, \theta) = (r\cos(\theta), r\sin(\theta), r)$$

- Try S_2 first:

$$\frac{\partial S_2}{\partial r} = (\cos(\theta), \sin(\theta), 1), \ \frac{\partial S_2}{\partial \theta} = (-r\sin(\theta), r\cos(\theta), 0).$$

  So

$$\begin{aligned} \frac{\partial S_2}{\partial r} \times \frac{\partial S_2}{\partial \theta} &= (0 - r\cos(\theta), -0 - r\sin(\theta), r\cos^2(\theta) + r\sin^2(\theta)) \\ &= (-r\cos(\theta), -r\sin(\theta), r) \end{aligned}$$

  That means

$$\left\| \frac{\partial S_2}{\partial r} \times \frac{\partial S_2}{\partial \theta} \right\| = \sqrt{r^2[\cos^2(\theta) + \sin^2(\theta)] + r^2} = r\sqrt{2}$$

- Hence the surface area is

$$\begin{aligned} \int_A dS &= \int_{(r,\theta):r\leq 1} r\sqrt{2} \ dr \ d\theta \\ &= \int_{\theta\in[0,2\pi]} \int_{r\in[0,1]} r\sqrt{2} \ drd\theta && \text{by Tonelli} \\ &= 2\pi r^2\sqrt{2}/2|_0^1 = \pi\sqrt{2} \approx \boxed{4.442}. \end{aligned}$$

What is the surface area of the unit sphere?

- Implicit equation of the sphere: $x^2 + y^2 + z^2 = 1$
- Use spherical coordinates to parameterize sphere:

$$\begin{aligned} x &= \sin(\phi)\cos(\theta) \\ y &= \sin(\phi)\sin(\theta) \\ z &= \cos(\phi) \\ S(\phi,\theta) &= (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)) \end{aligned}$$

- So that means:

$$\begin{aligned} \frac{\partial S}{\partial \phi} &= (\cos(\phi)\cos(\theta), \cos(\phi)\sin(\theta), -\sin(\phi)) \\ \frac{\partial S}{\partial \theta} &= (-\sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta), 0) \\ \frac{\partial S}{\partial \phi} \times \frac{\partial S}{\partial \theta} &= (+\sin^2(\phi)\cos(\theta), +\sin^2(\phi)\sin(\theta), \\ &\quad \cos(\phi)\sin(\phi)\cos^2(\theta) + \cos(\phi)\sin(\phi)\sin^2(\theta)) \\ &= \sin(\phi)(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)) \end{aligned}$$

Taking the norm then gives:

$$\begin{aligned} \left\| \frac{\partial S}{\partial \phi} \times \frac{\partial S}{\partial \theta} \right\| &= |\sin(\phi)| \sqrt{\sin^2(\phi)\cos^2(\theta) + \sin^2(\phi)\sin^2(\theta) + \cos^2(\phi)} \\ &= |\sin(\phi)| \sqrt{\sin^2(\phi) + \cos^2(\phi)} \\ &= |\sin(\phi)| \end{aligned}$$

Hence the surface area of the sphere is

$$\begin{aligned} S &= \int_{\theta\in[0,2\pi],\phi\in[0,\pi]} |\sin(\phi)|\ d(\theta,\phi) \\ &= \int_{\theta\in[0,2\pi]} \int_{\phi\in[0,\pi]} \sin(\phi)\ d\theta\ d\phi && \text{by Tonelli} \\ &= 2\pi(-\cos(\phi))|_0^\pi = 2\pi(-(-1) - (-1)) = 4\pi \approx \boxed{12.56}. \end{aligned}$$

34.1 Other applications of surface integrals

- So far, just done $\int_A \ dS$ to find the surface area of the sphere.
- Can also put an integrand into the integral.

Definition 102
Let $\psi(v)$ be the temperature of point v on a surface. Call $\int_A \psi\ ||dS||$ the **heat flux** of the surface.

Example

- For the surface $f(x,y) = x^2 + y$, the temperature at point (x,y) for $x \in [0,1]$ is $2x$. Find the heat flux over the surface for x from 0 to 1 and y from -1 to 1.
- Key as before is putting dS in terms of $dx\ dy$. Here

$$S(x,y) = (x, y, f(x,y)),$$

so

$$\frac{\partial S}{\partial x} = (1, 0, 2x), \ \frac{\partial S}{\partial y} = (0, 1, 1)$$
$$\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} = ((0)(1) - (2x)(1), (1)(1) - (2x)(0), (1)(1) - (0)(0))$$
$$= (-2x, 1, 1)$$
$$\left\| \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right\| = \sqrt{(-2x)^2 + (1)^2 + 1^2} = \sqrt{2 + 4x^2}$$

- So

$$\int_A 2x \; dS = \int_{x\in[0,1]} \int_{y\in[-1,1]} 2x\sqrt{2+4x^2} \; dy \; dx$$
$$= 2\int_{x\in[0,1]} 2x\sqrt{2+4x^2} \; dx$$
$$= 2(2+4x^2)^{3/2}/(3/2)/4|_0^1$$
$$= 2(1/6)[6^{3/2} - 1] \approx \boxed{4.565}$$

Actually have a shortcut that helps in cases like these:

Fact 58
If $S(x, y) = (x, y, f(x, y))$, then

$$\|dS\| = \left\| \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right\| \; dx \; dy = \sqrt{1^2 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \; dx \; dy.$$

- The proof just comes from working through the cross product.
- Compare to a fact learned in Calc I about arc length

Fact 59
If $C(t) = (t, f(t))$, then
$$||dC(t)|| = \|C'(t)\| \; dt = \sqrt{1 + (f'(t))^2} \; dt.$$

34.2 Manifolds

Idea

- A *manifold* is a collection of points that locally looks like a straight line (in 1-D) a plane (in 2-D) standard three dimensional space (in 3-D) and so on.
- On surface of Earth, when you look around locally, the surface appears to be a flat plane.
- For a point $(1/\sqrt{2}, 1/\sqrt{2})$ on the unit circle, locally the curve appears to be a flat line.
- Informally, you can bend the curve slightly in order to get a straight line locally. Curve is not too "crinkly".
- A curve embedded in 2-D or 3-D or higher dimensional space is a 1-manifold.
  - Circle, parabola, hyperbola, cubic
- A surface embedded in 3 (or higher) dimensional space is a 2-manifold.
  - Möebius strip

  - Klein bottle

- A solid sculpture in 4 (or higher) dimensional space is a 3-manifold.

Problems

34.1: Let S be the surface defined by the intersection of the half cylinder $\{(x, y, z) : x \geq 0, x^2 + y^2 \leq 9\}$ with the plane $z = (1/2)y$. If the temperature at point (x, y, z) on this surface is x, find the heat flux from the surface.

35 Curvature

Question of the Day What is the curvature of the curve $C(t) = (t, t^2)$ at $(2, 4)$?

Today

- Curvature
- Manifolds with low curvature are almost flat
- Manifolds with high curvature are very bendy
- Small circles (merry go round) very high curvature, strong acceleration as you travel along the curve
- Large circles (the Equator) very low curvature, low acceleration as you travel.
- How quickly speed changes gives curvature
- Curves with high curvature change quickly, low curvature change slowly
- Engineering materials tend to have a maximum curvature beyond which they break.

Definition 103
Let $C(t) \in C^2$. Let $T(t) = C'(t)/\|C'(t)\|$ be the normalized tangent vector. Let $s(t)$ be the arclength of the curve over times $[0, t]$. Then the **curvature** at a point is

$$\kappa = \left\| \frac{dT}{ds} \right\|.$$

- Usually use the Greek letter kappa, κ to represent curvature.

Fact 60
The curvature can also be found by

$$\kappa = \left\| \frac{dT}{dt} \right\| \cdot \frac{1}{\|C'(t)\|}.$$

This is known as the *two derivatives* form of curvature.

Proof. Recall that $ds = \|C'(t)\| \ dt$. So by the chain rule:

$$\frac{dT}{ds} = \frac{dT}{dt} \cdot \frac{dt}{ds} = \frac{dT}{dt} \cdot \frac{1}{\|C'(t)\|}.$$

□

Example

- What is the curvature of a circle of radius a?
- First must parameterize. Any way of parameterizing will do!

  $$C(t) = (a\cos(3t), a\sin(3t)).$$

  [Have constant of 3 to show that method of parameterizing doesn't matter.]
- Find $\|C'(t)\|$.

$$\begin{aligned}\|C'(t)\| &= \sqrt{(-3a\sin(3t))^2 + (3a\cos(3t))^2} \\ &= \sqrt{(3a)^2(\sin^2(3t) + \cos^2(3t))} \\ &= 3a\sqrt{1} = 3a.\end{aligned}$$

- Find $T(t)$:
$$T(t) = \frac{C'(t)}{\|C'(t)\|} = (\cos(3t), \sin(3t)).$$
[Note that $\|T(t)\|$ should always equal 1, which it does here.]

- Find κ:
$$\left\|\frac{dT}{dt}\right\| \cdot \frac{1}{\|C'(t)\|} = \frac{\|(-3\sin(3t), 3\cos(3t))\|}{3a} = \frac{3}{3a} = \boxed{\frac{1}{a}}$$

- The bigger radius the circle, the smaller the curvature.

Fact 61
The units of curvature are the inverse of the units of distance.

35.1 Curvature for 1-manifolds in 2-D

Qotd

- First need a parameterization, go with the easy one:
$$C(t) = (t, t^2).$$
Then $C'(t) = (1, 2t)$, so $\|C'(t)\| = \sqrt{1+4t^2}$.

- Let $r = \|C'(t)\| = \sqrt{1+4t^2}$ to make the calculations simpler. Then note $dr/dt = 8t(1/2)(1+4t^2)^{-1/2} = 4t/r$. So
$$T(t) = (1, 2t)/r = (1/r, 2t/r)$$
$$T'(t) = (-(1/r^2)(4t/r), 2/r - 2t(1/r^2)(4t/r))$$

- That means that
$$\begin{aligned}\frac{T'(t)}{\|C'(t)\|} = \frac{T'(t)}{r} &= (-4t/r^4, 2/r^2 - 8t^2/r^4)\\ &= (1/r^4)(-4t, 2r^2 - 8t^2)\\ &= (1/r^4)(-4t, 2 - 8t^2 + 8t^2)\\ &= (1/r^4)(-4t, 2)\end{aligned}$$
Taking the norm gives
$$\begin{aligned}\frac{\|T'(t)\|}{\|C'(t)\|} &= (1/r^4)\sqrt{(-4t)^2 + 2^2}\\ &= (1/r^4)\sqrt{16t^2+4}\\ &= (1/r^4)\sqrt{4}\cdot\sqrt{1+4t^2}\\ &= 2/r^{-3} = \boxed{2(1+4t^2)^{-3/2}}.\end{aligned}$$

Can generalize this calculation to get the following result:

Fact 62
Suppose $f(t) \in C^2$. Then for $C(t) = (t, f(t))$,
$$\kappa = \frac{|f''(t)|}{(1+f'(t)^2)^{3/2}}.$$

35.2 Coordinate free curvature

- Note denominator: for $C(t) = (t, f(t))$, $(1 + f'(t)^2)^{1/2} = \|C'(t)\|$.
- What determines curvature is how quickly the turn occurs for a car traveling along the curve.
- Physically, this is the acceleration perpendicular to the direction of travel.
- $\|C'(t) \times C''(t)\|$ captures this, but must be properly normalized.
- Working through the details gives the following relationship that can be useful for theory.

Fact 63
For $C(t) \in C^2$:

$$\kappa = \frac{\|C'(t) \times C''(t)\|}{\|C'(t)\|^3}.$$

This called the *coordinate free representation* of curvature.

- In physics terms, curvature is the area of the parallelogram spanned by velocity and acceleration, divided by the speed cubed.
- Might have heard that general relativity treats gravity as the curvature of spacetime.
- Can define the curvature of 2-manifolds, or n-manifolds as well, leads to useful mathematical tools for theoretical physics such as general relativity.

Problems

35.1: What is the curvature of a circle of radius 2?

36 The Divergence Theorem

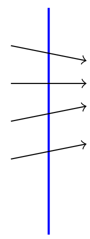
Question of the Day What is the flux of $F(x, y, z) = (x, y, z)$ over the surface of the sphere $x^2 + y^2 + z^2 = a^2$?

Today

- Flux
- The Divergence Theorem in 3D

36.1 Flux

- "Flux" is Latin for flow
- Flux is the flow across a curve or surface:



- Only part normal to curve/surface contributes to flux

$$\int_R F \cdot dN,$$

where dN is differential normal to the curve/surface.

- In general, hard to calculate. For $S(u, v)$,

$$dN = \frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}\, du\, dv.$$

Calculating flux by hand over a 2-manifold R

1: Parameterize R as $S(u, v)$

2: Calculate, $\partial S/\partial u$ and $\partial S/\partial v$

3: Take the cross product between $\partial S/\partial u$ and $\partial S/\partial v$

4: Take the dot product of this and F

5: Solve the two dimensional integral in u and v that results.

36.2 The Divergence Theorem in 3D

An easier way

- When R is a closed surface with an inside and an outside, things are much easier.
- Can use the higher dimensional version of the Divergence Theorem.

Theorem 13 (Divergence Theorem)
Let $F : \mathbb{R}^n \to \mathbb{R}^n$ and $S \subset \mathbb{R}^n$ have a closed boundary. Then

$$\int_{\partial S} F \cdot dN = \int_S \operatorname{div}(F)\, d\mathbb{R}^n,$$

where $\operatorname{div}(F) = \nabla \cdot F$.

Finding divergence

- Recall
$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right).$$
- For $F(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$,
$$\nabla \cdot F = \frac{\partial y_1}{\partial x_1} + \frac{\partial y_2}{\partial x_2} + \cdots + \frac{\partial y_n}{\partial x_n}.$$

Qotd

- Surface of a sphere is a closed 2-manifold
- Can apply Div. Thm!
- First, find the divergence of $F(x, y, z) = (x, y, z)$:
$$\operatorname{div}(F) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$$
- So by the Div Thm,
$$\text{flux} = \int_S 3\ dx\ dy\ dz = 3(4/3)\pi a^3 = 4\pi a^3.$$

The long way

- Sometimes, have to do problems the long way
- For instance, when surface is not closed

Qotd the long way

- Step 1: parameterize the sphere:
$$S(\phi, \theta) = (a \sin(\phi)\cos(\theta), a\sin(\phi)\sin(\theta), a\cos(\phi)).$$
- Step 2: Find $\partial S/\partial \phi$ and $\partial S \partial \theta$
$$\begin{aligned}\frac{\partial S}{\partial \phi} &= (a\cos(\phi)\cos(\theta), a\cos(\phi)\sin(\theta), -a\sin(\phi))\\ \frac{\partial S}{\partial \theta} &= (-a\sin(\phi)\sin(\theta), a\sin(\phi)\cos(\theta), 0)\end{aligned}$$
- Step 3: Find $s(\phi, \theta) = \partial S/\partial \phi \times \partial S \partial \theta$
$$\begin{aligned}s(\phi, \theta) = {} & (0(a\cos(\phi)\sin(\theta)) - (a\sin(\phi)\cos(\theta))(-a\sin(\phi)),\\ & -((a\cos(\phi)\sin(\theta))(0) - (-a\sin(\phi)\sin(\theta))(-a\sin(\phi)))\\ & (a\cos(\phi)\cos(\theta))(a\sin(\phi)\cos(\theta)) - (-a\cos(\phi)\sin(\theta))(-a\sin(\phi)\sin(\theta)))\\ = {} & (a^2\sin^2(\phi)\cos(\theta), a^2\sin^2(\phi)\sin(\theta), a^2\sin(\phi)\cos(\phi)[\cos^2(\theta) + \sin^2(\theta)])\\ = {} & a\sin(\phi)(x, y, z)\end{aligned}$$
- Step 4: Take $F(x, y, z) \cdot s(\phi, \theta)$
$$\begin{aligned}(x, y, z) \cdot a\sin(\phi)(x, y, z) &= a^2\sin(\theta)(x^2 + y^2 + z^2)\\ &= a^3\sin(\theta)\end{aligned}$$

- Step 5: Integrate over $(\phi, \theta) \in [0, \pi] \times [0, 2\pi]$. By Fubini can break into iterated integral, so:

$$\begin{aligned}\int_{\partial S} F \cdot dN &= \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} a^3 \sin(\phi)\ d\theta\ d\phi \\ &= a^3 \int_{\phi=0}^{\pi} 2\pi \sin(\phi)\ d\phi \\ &= 2\pi a^3 (-\cos(\phi))|_0^{2\pi} \\ &= 2\pi a^3 (-(-1) - (-1)) = 4\pi a^3.\end{aligned}$$

The right hand rule

- Note: $s(u, v) = -s(v, u)$
- So how do you know which order to do?
- Right hand rule: $s(u, v)$ put right hand, curl through u then v, thumb points in direction of $\partial S/\partial u \times \partial S/\partial v$.
- Positive ϕ runs from north pole to south pole on sphere, positive θ runs from west to east. Curl fingers on globe, thumb points outwards.

37 Differential Forms

Question of the Day What do the FTC, Generalized FTC, Green's Thm and the Divergence Theorem have in common?

Today

- Stokes' Theorem

In this course

- Fundamental Theorem of Calculus (FTC):

$$\int_{[a,b]} f'(x)\ dx = f(b) - f(a)$$

- Generalized FTC:

$$\int_{C([t_0,t_1])} \nabla\phi(x)\ dx = \phi(C(t_1)) - \phi(C(t_0))$$

- Green's Theorem

$$\int_{\partial A\ (\text{ccw})} F \cdot dC = \int_A \text{rot}(F)\ dx\ dy$$

- Divergence Theorem (2D)

$$\int_{\partial A} F \cdot dN = \int_A \text{div}(F)\ dx\ dy$$

- Divergence Theorem (higher dimensions)

$$\int_{\partial V} F \cdot dS = \int_V \text{div}(F)\ d\mathbb{R}^n$$

37.1 Stokes' Theorem

All those results are a special case of the following general result:

Theorem 14 (Stokes' Theorem)
Let Ω be an oriented smooth n-manifold and ω an n-differential form that is nonzero over a compact region. Then

$$\int_{\partial\Omega} \omega = \int_\Omega d\omega.$$

To understand this theorem, we need to understand what a differential form is. The first thing needed is the wedge product

37.2 Differential forms

There are 0-forms, 1-forms, 2-forms, etcetera...

Definition 104
For $U \in \mathbb{R}$ an open set, a function $f : U \to \mathbb{R}$ is a **0-form**. For $f \in C^1$, the differential of a 0-form is $d[f(x)] = f'(x)\ dx$.

Definition 105
A **1-form** is a 0-form times a differential, such as dx, dy, or dz. The sum of 1-forms is also a 1-form.

- Ex: $w = x^2y\ dx,\ w = x^2z\ dx + \exp(z)\ dy + z^2y\ dz$ are all 1-forms.
- Note that the differential of a 0-form is a 1-form

Definition 106
An **n-differential form** consists of a real valued function of n variables multiplied by n different differentials, or the sum of such objects.

- Ex: $w = x^2y\ dx\ dy + xz\ dx\ dz$ is a 2-form.
- Ex: $(x + y + z)\ dx\ dy\ dz$ is a 3-form.

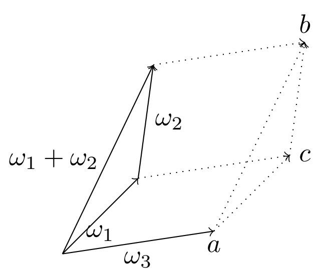
37.3 Wedge Product

- The *wedge product* gives us a way to multiply differential forms.
- Let ω_1 be an ℓ-form, ω_2 be a k-form. Then

$$\omega_3 = \omega_1 \wedge \omega_2$$

  is an $(\ell + k)$-form.

The first property of the wedge product is that like a regular product, it is distributive. The picture is as follows:



The area spanned by ω_1 and ω_3 plus the area spanned by ω_2 and ω_3 is the same as the area spanned by $\omega_1 + \omega_2$ and ω_3 (just shift by $-\omega_3$ the triangle abc). Mathematically, this means that we want

$$(\omega_1 \wedge \omega_3) + (\omega_2 \wedge \omega_3) = (\omega_1 + \omega_2) \wedge \omega_3.$$

The second property relates to the right hand rule. If we take the fingers of our right hand, and curl from ω_3 to ω_1, then the thumb points in the positive z direction. So $\omega_3 \wedge \omega_1$ is positive. However, if we start at ω_1 and curl our fingers of our right hand through ω_3, then our thumb points in the negative z direction. The area of the parallelogram that they span is the same though. That gives us the property that $\omega_1 \wedge \omega_3 = -\omega_3 \wedge \omega_1$. A generalized version of this property, together with the distributive and other properities of the wedge product, are given below.

Axioms for the wedge product

1: Zero rule: if $(\forall\omega)(\omega + 0 = \omega)$, then $\omega \wedge 0 = 0$.

2: Distributive: $(\omega_1 + \omega_2) \wedge \eta = (\omega_1 \wedge \eta) + (\omega_2 \wedge \eta)$.

3: Associative: For 1-forms ω and η, $\omega \wedge \eta = -\eta \wedge \omega$.

4: Anticommutative: Let ω be a k-form and η and ℓ-form. Then $\omega \wedge \eta = (-1)^{k\ell}(\eta \wedge \omega)$.

5: 0-forms are constants: For f a 0-form,

$$\omega \wedge (f\eta) = (f\omega) \wedge \eta = f(\omega \wedge \eta).$$

6: Basic laws

$$dx_1\ dx_2 \cdots dx_n \wedge dy = dx_1\ dx_2 \cdots dx_n\ dy.$$

and

$$dx \wedge dx = 0.$$

37.4 Differentiating forms

Here are the rules for finding $d\omega$ given ω:

1: If ω is an n-form, then $d\omega$ is an $(n+1)$-form.

2: Additive rule $d(\omega_1 + \omega_2) = d\omega_1 + d\omega 2$

3: Product rule $d(\omega \wedge \eta) = d\omega \wedge \eta) + (-1)^k(\omega \wedge d\eta)$.

4: Differential of something small is 0:

$$d(d\omega) = 0.$$

5: Connecting to regular differentiation: For $f(x_1, \dots, x_n)$ a 0-form,

$$df = \nabla f \cdot (dx_1, \dots, dx_n) = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

37.5 The Generalized Fundamental Theorem of Calculus

Consider a curve integral over a curve $C \in C^1$ that starts at a and moves along a path to b. Then consider

$$\int_{C:a \to b} \nabla\phi \cdot dC.$$

Writing $\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$, we have

$$\begin{aligned}
\int_{C:a \text{ to } b} &= \int_{C:a \text{ to } b} \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right) \cdot (dx, dy) \\
&= \int_{C:a \text{ to } b} \frac{\partial\phi}{\partial x}\ dx + \frac{\partial\phi}{\partial y}\ dy \\
&= \int_{C:a \text{ to } b} d\phi.
\end{aligned}$$

By Stokes' Theorem:

$$\int_{C:a \text{ to } b} d\phi = \int_{a \text{ and } b} \phi = \phi(b) - \phi(a).$$

37.6 Green's Theorem

- Recall $F(x,y) = (f_1(x,y), f_2(x,y))$, then

$$\int_{C([t_0,t_1])} F \cdot dC = \int_{t_0}^{t_1} f_1\ dx + f_2\ dy.$$

- Let $\omega = f_1\ dx + f_2\ dy$. That is a 1-form!

- Now to find $d\omega$:

$$
\begin{aligned}
d\omega &= d(f_1\ dx) + d(f_2\ dy) \\
&= d(f_1) \wedge dx + (-1)^0 f_1 \wedge d(dx) + d(f_2) \wedge dy + (-1)^0 f_1 \wedge d(dy) \\
&= d(f_1) \wedge dx + d(f_2) \wedge dy \\
&= \left[\frac{\partial f_1}{dx}\ dx + \frac{\partial f_1}{dy} dy\right] \wedge dx + \left[\frac{\partial f_2}{dx}\ dx + \frac{\partial f_2}{dy} dy\right] \wedge dy \\
&= \frac{\partial f_1}{dx}\ dx \wedge dx + \frac{\partial f_1}{dy} dy \wedge dx + \frac{\partial f_2}{dx}\ dx \wedge dy + \frac{\partial f_2}{dy} dy \wedge dy \\
&= \frac{\partial f_1}{dy} dy \wedge dx + \frac{\partial f_2}{dx}\ dx \wedge dy \\
&= \frac{\partial f_2}{dx}\ dx \wedge dy - \frac{\partial f_1}{dy} dx \wedge dy \\
&= \mathrm{rot}(F)\ dx\ dy.
\end{aligned}
$$

- So by Stokes' Theorem:

$$\int_A \mathrm{rot}(F)\ dx\ dy = \int_{\partial A} f_1\ dx + f_2\ dy.$$

37.7 2×2 Jacobian

Consider $F(x, y) = (u(x, y), v(x, y))$. Then

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy,$$

and

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy,$$

So

$$
\begin{aligned}
du \wedge dv &= \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right) \wedge \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy\right) \\
&= \left(\frac{\partial u}{\partial x} dx \wedge \frac{\partial v}{\partial x} dx\right) + \left(\frac{\partial u}{\partial x} dx \wedge \frac{\partial v}{\partial y} dy\right) + \left(\frac{\partial u}{\partial y} dy \wedge \frac{\partial v}{\partial x} dx\right) + \left(\frac{\partial u}{\partial y} dy \wedge \frac{\partial v}{\partial y} dy\right) \\
&= \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} (dx \wedge dx) + \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} (dx \wedge dy) + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} (dy \wedge dx) + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} (dy \wedge dy).
\end{aligned}
$$

Since $dx \wedge dx = dy \wedge dy = 0$,

$$
\begin{aligned}
du \wedge dv &= \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} (dx \wedge dy) - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} (dx \wedge dy) \\
&= \left[\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}\right] dx \wedge dy \\
&= \det(DF)\ dx \wedge dy.
\end{aligned}
$$

The last step uses the value that

$$DF = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}.$$

Now, it is important to note that if the limits for x and y are oriented so that both dx and dy are always positive, and after changing variables the limits for u and v and also oriented so that du and dv are always positive, then the areas spanned will always be positive, which is why this is usually written:

$$du\ dv = |\det(DF)|\ dx\ dy.$$

The 3 by 3 and higher level Jacobians can be calculated the same way.

Problems

37.1: Show the Divergence Theorem in 2D using Stokes' Theorem.

37.2: Show the Divergence Theorem in 3D using Stokes' Theorem.

A Problem Solutions

1.1: Show that $(0, 2)$ and $(3, 0)$ are perpendicular vectors

Solution The vector $(0, 2)$ has length 2 and $(3, 0)$ has length 3. The difference between them is the vector $(0, 2) - (3, 0) = (-3, 2)$ which has length $\sqrt{(-3)^2 + 2^2} = \sqrt{13}$. Since $2^2 + 3^2 = (\sqrt{13})^2$, by the Pythagorean Theorem the two vectors must be perpendicular.

1.2: Show that $(2, 3)$ and $(-6, 4)$ are perpendicular vectors.

Solution Let $v_1 = (2, 3)$ and $v_2 = (-6, 4)$. Then the length of these two vectors are $\|v_1\| = \sqrt{2^2 + 3^2} = \sqrt{13}$ and $\|v_2\| = \sqrt{(-6)^2 + 4^2} = \sqrt{52}$.

The length of the vector connecting point v_1 to v_2 is

$$\|v_1 - v_2\| = \sqrt{(2 - (-6))^2 + (3 - 4)^2} = \sqrt{64 + 1} = \sqrt{65}.$$

Since $(\sqrt{13})^2 + (\sqrt{52})^2 = (\sqrt{65})^2$, by the Pythagorean Theorem the two vectors must be perpendicular.

1.3: Set up the integral to find the arclength of along the following curves:

(a) $P(t) = (\cos(t), \sin(t))$, $0 \leq t \leq \tau$

(b) $P(t) = (t, t^2, t^3)$, $0 \leq t \leq 1$

(c) $P(t) = (\exp(t), t, \sqrt{t})$, $1 \leq t \leq 2$

(d) $P(t) = (1, 1/(1 + t))$, $0 \leq t \leq 10$

Solution To set up these integrals, use the fact that

$$ds = \|P'(t)\| \ dt,$$

where $\|P'(t)\|$ is the *speed* as the particle travels along the curve. For our examples:

(a) $P'(t) = (-\sin(t), \cos(t))$, $\|P'(t)\| = \sqrt{(-\sin(t))^2 + \cos(t)^2} = \sqrt{1} = 1$. So the integral is:

$$\boxed{\int_{t=0}^{\tau} 1 \ dt.}$$

(b) $P'(t) = (1, 2t, 3t^2)$, $\|P'(t)\| = \sqrt{1^2 + (2t)^2 + (3t^2)^2}$, so the integral is:

$$\boxed{\int_0^1 \sqrt{1 + 4t^2 + 9t^4} \ dt.}$$

(c) $P'(t) = (\exp(t), 1, 1/(2\sqrt{t}))$, $\|P'(t)\| = \sqrt{\exp(t)^2 + 1^2 + 1/(2\sqrt{t})^2}$, so the integral is

$$\boxed{\int_{t=1}^{2} \sqrt{\exp(2t) + 1 + 1/(4t)} \ dt.}$$

(d) $P'(t) = (0, -1/(1 + t)^2)$, $\|P'(t)\| = \sqrt{0^2 + (-1/(1 + t))^2}$. So

$$\boxed{\int_{t=0}^{10} \frac{1}{1 + t} \ dt.}$$

1.4: Use Wolfram Alpha to numerically solve the above integrals to 4 significant figures.

Solution

(a) Since $\tau = 2\pi$, this can be found in Wolfram Alpha using:

```
integral of 1 from 0 to 2*pi
```

which gives $\boxed{6.283}$

(b) Using

```
integral of sqrt(1+4t^2+9t^4) from 0 to 1
```

gives $\boxed{252.5}$

(c) Using

```
integral of sqrt(exp(2*t)+1+1/(4*t)) from 1 to 2
```

gives $\boxed{4.805.}$

(d) Using

```
integral of 1/(1+t) from 0 to 10
```

gives $\boxed{2.397}$

2.1: What is $(x, y) + (3, 4)$ written as a single vector?

Solution $\boxed{(x + 3, y + 4)}$

2.2: True or false: the vectors $(1, 2, 3)$ and $(3, 2, 1)$ are the same vector.

Solution $\boxed{\text{False!}}$ [With vectors (unlike sets), the order of the components does matter.]

2.3: The Euclidean norm of $(-5, 0)$ is what?

Solution $\boxed{5}$

2.4: List the points in $\{2, 3\} \times \{-1, 0, 1\}$.

Solution This is the Cartesian product, or direct product of the sets, and will be:

$$\boxed{\{(2, -1), (2, 0), (2, 1), (3, -1), (3, 0), (3, 1)\}}$$

2.5: Write the following sums of scaled vectors as a single vector:

(a) $(2, 3) + (-1, 4)$

(b) $(x, y) + (w, z)$

(c) $(2, 3) + 2(-1, 4)$

(d) $(x, y) + 2(w, z)$

Solution

(a) $(1, 7)$

(b) $(x + w, y + z)$

(c) $(0, 11)$

(d) $(x + 2w, y + 2z)$

2.6: Find $\|(3, -2, 0, 2)\|$.

Solution The norm of a vector is the square root of the sum of the squares of the entry of the vector, so

$$\|(3, -2, 0, 2)\| = \sqrt{3^2 + (-2)^2 + 0^2 + 2^2} = \boxed{4.123.}$$

3.1: What is $(x, y) \cdot (3, 4)$?

Solution $\boxed{3x + 4y}$

3.2: What is $(3)(3,-2)$?

Solution $\boxed{(9,-6)}$

3.3: What is $c(3,-2)$ where $c \in \mathbb{R}$?

Solution $\boxed{(3c,-2c)}$

3.4: What is the cosine of the angle between $(1,0)$ and $(0,1)$?

Solution $\boxed{0}$

3.5: If v and w are perpendicular, what is $v \cdot w$?

Solution $\boxed{0}$

3.6: (a) What is $(2,3) \cdot (-1,-1)$?

(b) What is $(1,0,-1) \cdot (7,3,4)$?

(c) What is $(x,y) \cdot (2,-2)$?

Solution

(a) $(2)(-1) + (3)(-1) = \boxed{-5}$

(b) $(1)(7) + (0)(3) + (-1)(4) = 1 + 0 - 4 = \boxed{-3}$

(c) $\boxed{2x - 2y}$

3.7: Find the angle between vectors $(2,3)$ and $(-1,-4)$.

Solution (Solution not included.)

4.1: True or false: Speed at a point is always a real number.

Solution $\boxed{\text{True.}}$

4.2: True or false: Let $f(x,y) = 2x + xy^2$. Then $f \in C^1$.

Solution $\boxed{\text{True.}}$

4.3: State whether or not the following parameterized curves are in C^1.

(a) $P(t) = (t, t^2, e^t)$

(b) $P(t) = (|t|, \sin(t))$

Solution

(a) $\boxed{\text{In } C^1.}$

(b) $\boxed{\text{Not in } C^1.}$

4.4: A particle moves along a trajectory so that at time t its location is $(t, t^2, \exp(t))$.

(a) What is its velocity at time $t = 1$?

(b) What is its acceleration at time $t = 1$?

(c) What is its speed at time $t = 1$?

Solution

(a) The velocity is just $P'(t)$, or

$$\boxed{(1, 2t, \exp(t)}$$

(b) The acceleration is just $P''(t)$, or

$$\boxed{(0, 2, \exp(t)}$$

(c) The speed is the norm of the velocity, so

$$\|P'(t)\| = \sqrt{1^2 + (2t)^2 + \exp(t)^2} = \boxed{\sqrt{1 + 4t^2 + \exp(2t)}}.$$

4.5: For $P(t) = (\sin(t), t^2)$, find the tangent line to P at $t = 0$.

Solution First find $P'(t) = (\cos(t), 2t)$. Then the equation of the tangent line is

$$P_1(t) = P(t_0) + P'(t_0)(t - t_0).$$

Here $t_0 = 0$, so

$$P_1(t) = (0, 0) + (1, 0)(t - 0) \Rightarrow \boxed{P_1(t) = (t, 0)}$$

is the tangent line.

4.6: Suppose that a particle has a circular path parameterized by

$$C(t) = ((3\sqrt{3}/2)\sin(t), ((3\sqrt{3}/2)\cos(t)).$$

(a) Find the velocity of the particle at $t = \tau/4$.
(b) Find the speed of the particle at $t = \tau/4$.
(c) Find the acceleration of the particle at $t = \tau/4$.
(d) Write the equation of the tangent line to the path at $t = \tau/4$.

Solution

(a) The velocity is just $C'(t)$:

$$C'(t) = ((3\sqrt{3}/2)\cos(t), -(3\sqrt{3}/2)\sin(t)), \boxed{C'(\tau/4) = (0, -3\sqrt{3}/2)}$$

(b) The speed is the norm of the velocity:

$$\|C'(\tau/4)\| = \sqrt{0^2 + (-3\sqrt{3}/2)^2} = \boxed{3\sqrt{3}/2 \approx 2.598}.$$

(c) The acceleration is the derivative of the velocity:

$$C''(t) = (-(3\sqrt{3}/2)\sin(t), -(3\sqrt{3}/2)\cos(t)), \boxed{C''(\tau/4) = (-3\sqrt{3}/2, 0)}.$$

(d) The tangent line is

$$f_1(t) = C(\tau/4) + C'(\tau/4)(t - \tau/4) = \frac{3\sqrt{3}}{2}\left[(1, 0) + (0, 1)(t - \tau/4)\right]$$

so the tangent line is

$$\boxed{\frac{3\sqrt{3}}{2}(1, t - \tau/4)}.$$

5.1: Graph the level sets of $x = (1/2)y^2$.

Solution (Solution not included.)

5.2: Find the following partial derivatives.

(a) $\partial(x^2y)/\partial x$.
(b) $\partial(x^2y)/\partial y$.
(c) $\partial(x^2y)/\partial z$.
(d) $\partial(\exp(-2x))/\partial x$.

(e) $\partial(r/w)/\partial r$.

Solution

(a) $\partial(x^2y)/\partial x = \boxed{y}$.

(b) $\partial(x^2y)/\partial y = \boxed{x^2}$.

(c) $\partial(x^2y)/\partial z = \boxed{0}$.

(d) $\partial(\exp(-2x))/\partial x = \boxed{-2\exp(-2x)}$.

(e) $\partial(r/w)/\partial r = \boxed{1/w}$.

5.3: Find the following partial derivatives.

(a) $\partial[x^2y + 2y]/\partial y$.

(b) $\partial[x^2y + 2y]/\partial x$.

(c) $\partial[x^2y + 2y]/\partial z$.

Solution

(a) $\boxed{x^2 + 2.}$

(b) $\boxed{y.}$

(c) $\boxed{0.}$

6.1: Prove the following:

$$(\exists x)(2x = 10)$$

Solution.

Proof. Let $x = 5$. Then $2x = 10$. □

6.2: Prove that $\lim_{(x,y)\to(0,0)} 1 - x + y = 0$.

Solution. (Solution not included.)

7.1: (a) Find the best linear approximation for $f(x, y) = \sin(x + 2y)$ near $(\pi, 0)$.

(b) Use your approximation to estimate $\sin(x + 2y)$ at $(x, y) = (\pi + 0.1, 0.1)$.

Solution.

(a) The best linear approximation of f is

$$f_1((x_0, y_0) + (h_x, h_y) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (h_x, h_y).$$

Here $(x_0, y_0) = (\pi, 0)$ and $f(x_0, y_0) = \sin(\pi) = 0$. Since $\nabla \sin(x+2y) = (\cos(x+2y), 2\cos(x+2y))$, $\nabla \sin(x + 2y)|_{(x,y)=(\pi,0)} = (-1, -2)$, and the approximation is

$$\boxed{f_1((x_0, y_0) + (h_x, h_y)) = -h_x - 2h_y}.$$

(b) Here $h_x = h_y = 0.1$, so the approximation is $\boxed{\text{-0.3000}}$.

This approximation was done in terms of (h_x, h_y), how far away we move from the point $(\pi, 0)$. It can also of course, be entirely done in terms of x and y by using $(h_x, h_y) = (x - x_0, y - y_0)$.

(a) The best linear approximation of f is

$$f_1(x, y) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (x - \pi, y - 0).$$

Using $f(\pi, 0) = 0$ and $\nabla f(\pi, 0) = (-1, -2)$ from before,

$$\boxed{f_1(x, y) = -x - 2y + \pi.}$$

(b) Pluggin in $x = \pi + 0.1$ and $y = 0.1$ gives $f_1(\pi + 0.1, 0.1) = \boxed{\text{-0.3000}}$ as before.

8.1: Are the following sets of points written as explicit functions or as implicit functions?

(a) $y = 2x + 3$
(b) $x^2 + y^2 = 4$
(c) $z = x\exp(-xy)$
(d) $x\exp(-xy) - z = 0$

Solution.

(a) $\boxed{\text{Explict}}$

(b) $\boxed{\text{Implicit}}$

(c) $\boxed{\text{Explicit}}$

(d) $\boxed{\text{Implicit}}$

8.2: Find the tangent plane to $x^2 + y^2 + 2z^2 = 7$ at the point $(1, 2, 1)$

Solution. (Solution not included.)

8.3: Find the tangent line to $x^3 - y^2 = -1$ at the point $(2, 3)$.

Solution. (Solution not included.)

8.4: Find the directional derivative of $f(x, y) = (x^2, \exp(y))$ in the direction $(1, -1)$ from point $(2, 0)$.

Solution. (Solution not included.)

9.1: What is $\Delta(x^2 + y^2)$?

Solution. Since $(\partial/\partial x)(x^2+y^2) = 2x$, $(\partial^2/\partial x^2)(x^2+y^2) = (\partial/\partial x)(2x) = 2$. Similarly, $(\partial^2/\partial y^2)(x^2 + y^2) = 2$ as well, so

$$\Delta(x^2 + y^2) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right](x^2 + y^2) = 2 + 2 = \boxed{4}.$$

9.2: Be sure to justify your answers.

(a) Is $f(x, y, z) = (x^2 + y^2 + 2z^2)^{-1}$ rotationally symmetric?
(b) Is $f(x, y, z) = (x^2 + y^2 + z^2)^{-1}$ rotationally symmetric?
(c) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ can be written as $f(v) = 1/\|v\|^2$. Find the gradient of f.

Solution. (Solution not provided.)

10.1: Which of the following sets are bounded? (You do not have to prove your answer.)

(a) $[4, \infty)$
(b) $(-\infty, \infty)$
(c) $[0, 3)$

Solution.

(a) $\boxed{\text{unbounded}}$ The set of numbers goes off to infinity.

(b) $\boxed{\text{unbounded}}$ This one goes off to infinity in both directions!

(c) $\boxed{\text{bounded}}$ Any point in the set is at most distance 3 from the origin.

10.2: Suppose $f(x) = x^3 - x$.

(a) Find $\max_{x\in[-1,2]} f(x)$.

(b) Find $\arg\max_{x \in [-1,2]} f(x)$.

(c) Find $\min_{x \in [-1,2]} f(x)$.

(d) Find $\arg\min_{x \in [-1,2]} f(x)$.

Solution. (Solution not included.)

11.1: Which of the following sets are bounded? (You do not have to prove your answer.)

(a) $\{(x, y) : x^2 + 2y^2 \le 4\}$

(b) $\{(x, y) : x^2 \ge y\}$

(c) $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$

Solution.

(a) $\boxed{\text{bounded}}$ Any point in the set is distance at most 2 from the origin.

(b) $\boxed{\text{unbounded}}$ For any y value, x can be anything from y^2 up to ∞, so infinitely far from the origin.

(c) $\boxed{\text{bounded}}$ Every point in the set is exactly distance 1 from the origin.

11.2: Suppose $f(x, y) = \exp(-x^2 - 2y^2)$. Let $A = \{(x, y) : x^2 + y^2 \le 4\}$.

(a) Find $\max_A f(x, y)$.

(b) Find $\arg\max_A f(x, y)$.

(c) Find $\min_A f(x, y)$.

(d) Find $\arg\min_A f(x, y)$.

Solution. (Solution not included.)

12.1: (a) What is $\max_{x \in [0,\infty)} x^2 \exp(-2x)$?

(b) What is $\max_{x \in (-\infty,\infty)} 3 - x^2$?

(c) What is $\min_{x \in (-\infty,\infty)} |x|$?

Solution.

(a) Let $f(x) = x^2 \exp(-x)$. Then

$$f'(x) = 2x \exp(-2x) + x^2(-\exp(-2x)) = \exp(-2x)x(2 - x).$$

Since $\exp(-2x)$ and x are nonnegative in $[0, \infty)$, $f'(x) \ge 0$ for $2 - x \ge 0$ (so $x \le 2$) and $f'(x) \le 0$ for $2 - x \le 0$ (so $x \ge 2$). Hence

$$\max_{x \in [0,2]} x^2 \exp(-x) = 2^2 \exp(-2) \text{ and } \max_{x \in [2,\infty)} x^2 \exp(-x) = 2^2 \exp(-2).$$

Putting this together gives

$$\max_{x \in [0,\infty)} x^2 \exp(-x) = 4e^{-2} \approx \boxed{0.5413}.$$

(b) Let $f(x) = 3 - x^2$. Then $f'(x) = -2x$, so $f'(x) \ge 0 \Leftrightarrow x \le 0$ and $f'(x) \le 0 \Leftrightarrow x \ge 0$. Hence

$$\max_{x \in [0,\infty)} 3 - x^2 = 3,$$

and

$$\max_{x \in (-\infty,0]} 3 - x^2 = 3.$$

Taken together, $\max_{x \in (-\infty,\infty)} = \boxed{3}$.

(c) For $x \in [0, \infty)$, $f(x) = x$ and $f'(x) = 1 \geq 0$. For $x \in (-\infty, 0]$, $f(x) = -x$ and $f'(x) = -1 \leq 0$. Hence

$$\min_{x\in[0,\infty)} |x| = \min_{x\in[0,\infty)} x = 0,$$

and

$$\min_{x\in(-\infty,0]} |x| = \min_{x\in(-\infty,0]} -x = 0.$$

Taken together, these give

$$\min_{x\in(-\infty,\infty)} |x| = \boxed{0}.$$

13.1: Solve the following optimzation problems.

(a) Find $\max\{x + y^2 | 2x^2 + y^2 \leq 2\}$.

(b) Find $\min\{x + y^2 | 2x^2 + y^2 \leq 2\}$.

(c) Find $\arg\max\{x + y^2 | 2x^2 + y^2 \leq 2\}$.

Solution. Let $A = \{(x, y) : 2x^2 + y^2 \leq 2\}$. Then A is closed and bounded, and so is compact. First find any critical points in $\text{int}(A)$ by setting ∇f to the zero vector.

Hence $f(x, y) = x + y^2$, and so $\nabla f(x, y) = (1, 2y)$ which never equals $(0, 0)$ so there are no critical points in the interior.

Now $\text{bdy}(A) = \{(x, y) : 2x^2 + y^2 = 2\}$, and $g(x, y) = 2x^2 + y^2$ and both g and f are C^1 functions, so the remaining critical points are when $\nabla f = \lambda \nabla g$ for some nonzero constant λ.

$$\nabla(x + y^2) = \lambda\nabla(2x^2 + y^2) \Rightarrow (1, 2y) = \lambda(4x, 2y),$$

so the three equations we need to solve are:

$$\begin{aligned} 1 &= \lambda \cdot 4x \\ 2y &= \lambda(2y) \\ 2x^2 + y^2 &= 2. \end{aligned}$$

If $y = 0$, then $2x^2 = 2$, $x^2 = 1$ and $x \in \{-1, 1\}$. So two possibilities are $\{(-1, 0), (1, 0)\}$.

If $y \neq 0$, then dividing through the second equation by $2y$ gives $\lambda = 1$, so $4x = 1$ and $x = 1/4$. Then $y^2 = 2 - 2(1/4)^2$ and $y \in \{\sqrt{15/8}, -\sqrt{15/8}\}$.

Filling out the table of function evaluations gives:

(x, y)	$x + y^2$
$(-1, 0)$	-1
$(1, 0)$	1
$(1/4, \sqrt{15/8})$	$17/8 = 2.125$
$(1/4, -\sqrt{15/8})$	$17/8 = 2.125$

Now use the extreme value theorem to get the following.

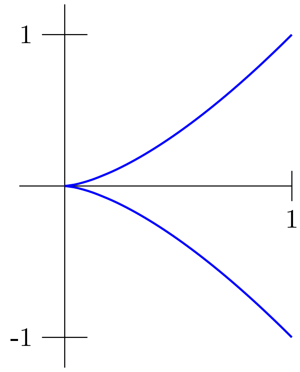
(a) $\boxed{2.125}$

(b) $\boxed{-1}$

(c) $\boxed{\{(0.2500, 1.369), (0.2500, -1.369)\}}$

14.1: Graph $\{(x, y) : x^3 - y^2 = 0, x \in [-1, 1]\}$

Solution. Note that if $x^3 - y^2 = 0$, then $x^3 = y^2$. Since $y^2 \geq 0$, that means $x^3 \geq 0$, so really we need only worry about $x \in [0, 1]$. Taking the square root of both sides of $x^3 = y^2$ gives $x^{3/2} = |y|$. Hence the graph looks like:



14.2: (a) Find $\max\{x^2+y|x^3-y^2=0, x\in[-1,1]\}$

(b) Find $\arg\max\{x^2+y|x^3-y^2=0, x\in[-1,1]\}$

(c) Find $\min\{x^2+y|x^3-y^2=0, x\in[-1,1]\}$

(d) Find $\arg\min\{x^2+y|x^3-y^2=0, x\in[-1,1]\}$

Solution. Here $f(x,y)=x^2+y$ and $g(x,y)=x^3-y^2$. All points lie on the boundary. The function $f(x,y)\in C^1$ everywhere, but $g(x,y)\in C^1$ except at $(0,0)$, $(1,1)$ and $(1,-1)$, so those points have to be dealt with separately.

Where $g(x,y)$ is C^1, we have

$$\nabla g=(3x^2,-2y).$$

Hence $\nabla f=(2x,1)=\lambda(3x^2,-2y)$ together with $g(x,y)=0$ gives the following three equations:

$$\begin{aligned} 2x &= \lambda\cdot 3x^2 \\ 1 &= \lambda\cdot\lambda(-2y) \\ x^3-y^2 &= 0. \end{aligned}$$

If $x=0$ then $y=0$. If $x\neq 0$, then $\lambda=(2/3)(1/x)$. It cannot be the case that $y=0$, so $\lambda=-1/(2y)$. Hence

$$\frac{2}{3x}=-\frac{1}{2y}\Rightarrow y=\frac{3}{4}x.$$

Plugging that into $x^3-y^2=0$ gives

$$x^3-(9/16)x^2=x^2(x-9/16)=0,$$

so either $x=0$ (and again $y=0$ in that case) or $x=9/16$, and $y^2=(9/16)^3$ so $y=\pm(9/16)^{3/2}=27/64$.

Making a table of function values gives:

(x,y)	x^2+y
$(0,0)$	0
$(1,1)$	2
$(1,-1)$	0
$(9/16,27/64)$	$189/256=0.7382\ldots$
$(9/16,-27/64)$	$-27/256=-0.1054\ldots.$

With this table we can answer all parts of the question.

(a) The maximum value is $\boxed{2}$.

(b) The argument that maximizes the function is $\boxed{\{(1,1)\}}$.

(c) The minimum value of the function is $\boxed{-0.1054}$.

(d) The argument that minimizes the function is the one point $\{(9/16, -27/64) = \boxed{(0.5625, -.4218\ldots)}$.

14.3: (a) Find $\max_{x^2-1\leq y\leq 1-x^2} x^2 - 3y^2$

(b) Find $\min_{x^2-1\leq y\leq 1-x^2} x^2 - 3y^2$

Solution. (Solution not provided.)

15.1: Multiply the following row vectors times column vectors:

(a) $\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(b) $\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Solution

(a) $-3 + 2 = \boxed{-1}$

(b) $\boxed{3x + 2y}$

15.2: Calculate the following products of matrices.

(a) $\begin{pmatrix} 2 & 4 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & -2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Solution

(a) $\boxed{\begin{pmatrix} 4 & 2 \\ -1 & -1 \end{pmatrix}}$

(b) $\boxed{\begin{pmatrix} 4 & -2 & 4 \\ 6 & 2 & 6 \end{pmatrix}}$

15.3: What is $\begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}^2$?

Solution $\frac{1}{36} \begin{pmatrix} 15 & 21 \\ 14 & 22 \end{pmatrix} \approx \boxed{\begin{pmatrix} 0.4166 & 0.5833 \\ 0.3888 & 0.6111 \end{pmatrix}}$

16.1: Let $f(x, y) = \sin(x + 2y)$. Find the Hessian of f.

Solution This is the matrix of second partial derivatives. First take the first derivatives:

$$\nabla f(x, y) = \begin{pmatrix} \cos(x + 2y) & 2\cos(x + 2y) \end{pmatrix}$$

Now take the second derivatives to get the Hessian. For instance,

$$\frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} \sin(x + 2y) = \frac{\partial}{\partial x} 2\cos(x + 2y) = -2\sin(x + 2y).$$

Doing this for all four entries of the matrix gives

$$Hf(x, y) = \begin{pmatrix} -\sin(x + 2y) & -2\sin(x + 2y) \\ -2\sin(x + 2y) & -4\sin(x + 2y) \end{pmatrix}.$$

16.2: Continuing the last problem, find the second order Taylor approximation to f around the point $(\pi/2, 0)$.

Solution First we have $f(\pi/2, 0) = 1$, $\nabla f(0,0) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ and

$$Hf(0,0) = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}.$$

Hence

$$\begin{aligned} f_2(x,y) &= f(\pi/2,0) + \nabla f(\pi/2,0)\begin{pmatrix} x-\pi/2 \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x-\pi/2 & y \end{pmatrix} Hf(\pi/2,0)\begin{pmatrix} x-\pi/2 \\ y \end{pmatrix} \\ &= 1 + (1/2)[(x-\pi/2)^2(-1) + (x-\pi/2)(-2)(y) + (y)(-2)(x-\pi/2) + (-4)(y)^2 \\ &= 1 - x^2/2 - (\pi/2)x - \pi^2/4 - 2xy + \pi y - 2xy + \pi y - 4y^2 \\ &= \boxed{1 - \pi^2/8 - x^2/2 + (\pi/2)x + \pi y - 2xy - 2y^2.} \end{aligned}$$

16.3: Suppose $f(x,y) = \cos(x+2y)$.

(a) Find the gradient of f.
(b) Find the Hessian of f.
(c) What is $f_2(x,y)$, the second order approximation to f at $(x,y) = (0,0)$?
(d) In what direction should one move from $(\tau/4, \tau/4)$ in order to increase the value of f as quickly as possible?

Solution

(a) This is $\boxed{\nabla f = (-\sin(x+2y), -2\sin(x+2y))}$.

(b) Take the derivatve again to get:

$$Hf = D(\nabla f) = \boxed{\begin{pmatrix} -\cos(x+2y) & -2\cos(x+2y) \\ -2\cos(x+2y) & -4\cos(x+2y) \end{pmatrix}}.$$

(c) The 2nd order approximation is $f_2(x,y) = f_2(0,0) + (x,y)\nabla f(0,0) + \frac{1}{2}(x,y)Hf(0,0)\begin{pmatrix} x \\ y \end{pmatrix}$, which becomes in this case:

$$f_2(x,y) = 1 + \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

which simplifies to

$$\boxed{1 - (1/2)x^2 - 2xy - 2y^2}.$$

(d) This is just the gradient evaluated at $(\tau/4, \tau/4)$, so $\boxed{(1,2)}$.

17.1: Are the following matrices positive definite, negative definite, or neither?

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 3 \\ 3 & 2 \end{pmatrix}$

Solution

(a) Note that

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a^2 + b^2,$$

so as long as a and b are not both 0, this must be positive so the matrix is $\boxed{\text{positive definite.}}$ Of course, we could also apply our test: $1 > 0$ and $0^2 - (1)(1) < 0$, so the matrix is positive definite.

(b) This is the negative of the previous matrix, so must be $\boxed{\text{negative definite.}}$

(c) Here $3 > 0$ and $2^2 - (3)(2) = -2 < 0$, so the matrix is $\boxed{\text{positive definite.}}$

(d) Here $3 > 0$ so it cannot be negative definite, but $3^2 - (3)(2) = 3 > 0$, so it is not positive definite either. The matrix is $\boxed{\text{neither.}}$

18.1: State whether or not Tonelli's Theorem, compact Fubini's Theorem, both, or neither apply to the following integrals.

(a) $\int_{(x,y)\in\mathbb{R}^2} x^2 + y^2 \; d\mathbb{R}^2$.

(b) $\int_{(x,y)\in[0,2]\times[-1,1]} x + y \; d\mathbb{R}^2$.

(c) $\int_{(x,y)\in[0,2]\times[-1,1]} x + |y| \; d\mathbb{R}^2$.

(d) $\int_{(x,y)\in[0,\infty)\times[0,\infty)} 10 - (x-2)^2 + y^2 \; d\mathbb{R}^2$.

Solution

(a) $\boxed{\text{Tonelli.}}$ The function is nonnegative, so Tonelli's applies. Compact Fubini does not apply, since the region is unbounded.

(b) $\boxed{\text{Fubini.}}$ The function is continuous and $[0,2]\times[-1,1]$ is compact, so Fubini does apply. The function is negative in some places (such as $(0.3, 0.5)$) and positive in others (such as $(0.7, 0.5)$), so Tonelli can not be applied.

(c) $\boxed{\text{Both.}}$ Here the limits of integration are compact as in the last example, but now the function is nonnegative, so both Fubini and Tonelli can be used.

(d) $\boxed{\text{Neither.}}$ The limits of integration are unbounded, and there are places where the function is both positive and negative (such as $(2,1)$ and $(10,3)$).

18.2: Calculate the following integral:

$$I = \int_{(x,y)\in[0,\pi/3]\times[0,2\pi/3]} \sin(x + 2y) \; d\mathbb{R}^2.$$

Solution For $A = [0,\pi/3]\times[0,2\pi/3]$, there are places where $f(x,y) < 0$, for instance $f(\pi/4, 0) = \sqrt{2}/2 > 0$ but $f(\pi/4, \pi/2) = \sin(5\pi/4) = -\sqrt{2}2 < 0$. So Tonelli cannot be used.

However, the region $[0,\pi/3]\times[0,\pi]$ is closed and bounded, and so compact, and $\sin(x+2y)$ is continuous, so the compact Fubini theorem can be applied to give:

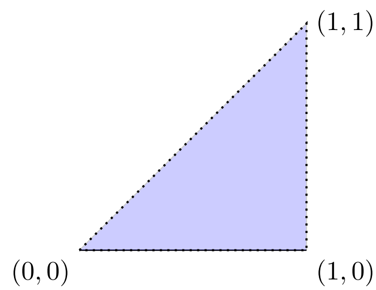
$$\begin{aligned}
\int_A \sin(x+2y)\; dA &= \int_{x=0}^{\pi/3} \int_{y=0}^{2\pi/3} \sin(x+2y)\; dy\; dx \\
&= \int_{x=0}^{\pi/3} -(1/2)\cos(x+2y)|_0^{2\pi/3}\; dx \\
&= \int_{x=0}^{\pi/3} (1/2)[\cos(x) + \cos(x + 4\pi/3)]\; dx \\
&= (1/2)[\sin(x) + \sin(x + 4\pi/3)]|_0^{\pi/3} \\
&= (1/2)[\sin(4\pi/3) + \sin(8\pi/3) - (\sin(0) + \sin(4\pi/3)] \\
&= (1/2)[\sqrt{3}/2] \approx \boxed{0.4330.}
\end{aligned}$$

19.1: Let B be the region strictly inside the triangle connecting the points $(0,0)$, $(1,0)$, and $(1,1)$. Find

$$\int_B x^{-3/2}\ d\mathbb{R}^2$$

or show that it does not converge.

Solution The graph of B looks like:



This triangle can be described using three inequalities:

$$y \geq 0, x \leq 1, x \leq y.$$

Since the region inside the triangle is open, the region is not compact, so we cannot invoke the compact Fubini condition.

However, $x^{-3/2}$ is always nonnegative inside the triangle, so Tonelli can be applied to state

$$\begin{aligned}\int_B x^{-3/2}\ d\mathbb{R}^2 &= \int_{x=0}^{1}\int_{y=0}^{x} x^{-3/2}\ dy\ dx \\ &= \int_{x=0}^{1} x^{-3/2} y|_0^x\ dx \qquad = \int_{x=0}^{1} x^{-1/2}\ dx \\ &= x^{1/2}/(1/2)|_0^1 \\ &= \boxed{2}.\end{aligned}$$

19.2: Find

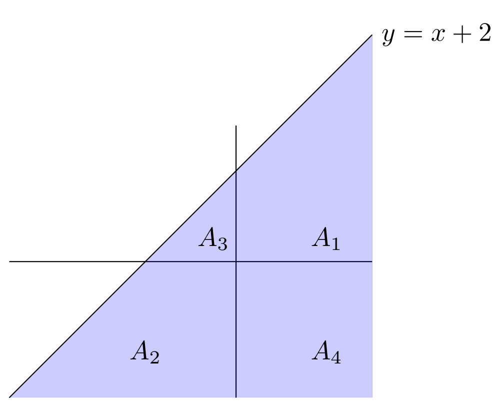
$$\int_{y\leq x+2} \frac{xy}{(1+x^2)(1+y^2)}\ d\mathbb{R}^2,$$

or show that it does not converge.

Solution The region $\{(x,y) : y \leq x+2\}$ can be written as

$$A_1 \cup A_2 \cup A_3 \cup A_4,$$

where



Note f is nonnegative on A_1 and A_2, and nonpositive on A_3 and A_4. So we can use Tonelli on each of these regions.

So

$$\begin{aligned}\int_{A_1} \frac{xy}{(1+x^2)(1+y^2)}\, d\mathbb{R}^2 &= \int_{x=0}^{\infty}\int_{y=0}^{x+2} \frac{xy}{(1+x^2)(1+y^2)}\, d\mathbb{R}^2 \\ &= \int_{x=0}^{\infty} \frac{x(1/2)\ln(1+y^2)}{1+x^2}\Big|_0^{x^2}\, d\mathbb{R}^2 \\ &= \int_{x=0}^{\infty} \frac{1}{2}\frac{x\ln(1+(x+2)^2)}{1+x^2}\, dx\end{aligned}$$

Now note that $x\ln(1+(x+2)^2)/(1+x^2) > x/(1+x^2)$, and

$$\int_{x=0}^{\infty} \frac{x}{1+x^2} = (1/2)\ln(1+x^2)|_0^{\infty}) = \infty,$$

so the integral over A_1 is infinity.

Trying the same thing over A_4, we get:

$$\begin{aligned}\int_{A_4} \frac{xy}{(1+x^2)(1+y^2)}\, d\mathbb{R}^2 &= \int_{x=0}^{\infty}\int_{y=-\infty}^{0} \frac{xy}{(1+x^2)(1+y^2)}\, d\mathbb{R}^2 \\ &= \int_{x=0}^{\infty} \frac{x(1/2)\ln(1+y^2)}{1+x^2}\Big|_0^{x^2}\, d\mathbb{R}^2\end{aligned}$$

The inside limit of the y integral is $-\infty$ regardless of the value of x, so the overall integral is $-\infty$. Since the integral over A_1 is ∞ and over A_2 is $-\infty$, then the overall integral over $A_1 \cup A_2 \cup A_3 \cup A_4$ $\boxed{\text{does not converge.}}$

20.1: Find $r(A)$ for the following examples:

(a) $A = (0,3) \times (4,8)$.
(b) $A = ((0,3) \times (4,8)) \cup ((-1,1) \times (-2,2))$.
(c) $A = (0,3) \times (4,8) \times (10,12)$.
(d) $A = [0,3] \times [4,8] \times [10,12]$.

Solution

(a) This is a single rectangle with area $(3-0)(8-4) = 3 \cdot 4 = \boxed{12}$.
(b) This is a pair of rectangles with areas $3 \cdot 4 = 12$ and $(1-(-1))(2-(-2)) = 2 \cdot 4 = 8$. Then $r(A)$ is the sum $\boxed{20}$.
(c) A is now a three dimensional box, the volume is the product of the side lengths: $(3-0)(8-4)(12-10) = \boxed{24}$.
(d) This is the same as the last problem but with closed intervals rather than open intervals, but that does not change the volume, so it is still $\boxed{24}$.

21.1: Find

$$\int_{x^2+y^2\le 9} \frac{1}{(x^2+y^2)^{1/4}}\, d\mathbb{R}^2.$$

Solution Transform to polar coordinates. Remember LID for limits, integrand, and differential.

First change the limits:

$$\int_{r^2\le 9,\theta\in[0,\tau]} \frac{1}{(x^2+y^2)^{1/4}}\, d\mathbb{R}^2,$$

next change the integrand:

$$\int_{r^2\leq 9,\theta\in[0,\tau]} \frac{1}{(r^2)^{1/4}}\ d\mathbb{R}^2,$$

and lastly change the differential:

$$\int_{r^2\leq 9,\theta\in[0,\tau]} \frac{1}{r^{1/2}} r\ dr\ d\theta.$$

Simplify

$$\int_{r\in[0,3],\theta\in[0,\tau]} r^{1/2}\ dr\ d\theta.$$

We can use Tonelli because the integrand is nonnegative or Fubini because the limits are closed and bounded and the integrand is continuous to write the integral as an iterated integral and solve.

$$\begin{aligned}\int_{r\in[0,3],\theta\in[0,\tau]} r^{1/2}\ dr\ d\theta &= \int_{r=0}^{3}\int_{\theta=0}^{\tau} r^{1/2}\ d\theta\ dr\\ &= \int_{r=0}^{3} \tau r^{1/2}\ dr\\ &= \tau r^{3/2}/(3/2)|_0^3\\ &= \tau 2\sqrt{3} \approx \boxed{21.77.}\end{aligned}$$

31.1: Suppose $H(r,\theta) = (r\cos(\theta), r\sin(\theta))$ and $G(x,y) = (xy, x^2+y^2)$.

(a) What is

$$[G\circ H](r,\theta)?$$

(b) Find $D[G\circ H]$ directly.

(c) Find $D[G\circ H]$ using the chain rule.

Solution

(a)

$$\begin{aligned}G(H(r,\theta) &= G(r\cos(\theta), r\sin(\theta))\\ &= (r^2\sin(\theta)\cos(\theta), r^2[\sin^2(\theta)+\cos^2(\theta)])\\ &= \boxed{(r^2\sin(\theta)\cos(\theta), r^2)}\end{aligned}$$

(b) Remember for the derivative of a vector field, the outputs go on the rows and the inputs on the columns:

$$\boxed{D[G\circ H](r,\theta) = \begin{matrix} \\ r^2\sin(\theta)\cos(\theta)\\ r^2\end{matrix}\begin{matrix} r & \theta \\ \left(\begin{matrix} 2r\sin(\theta)\cos(\theta) \\ 2r\end{matrix}\right. & \left.\begin{matrix} r^2[\cos^2(\theta)-\sin^2(\theta)] \\ 0\end{matrix}\right)\end{matrix}}$$

(c) Now let's use the chain rule. First we need DH and DG:

$$DH = \begin{matrix} \\ r\cos(\theta)\\ r\sin(\theta)\end{matrix}\begin{matrix} r & \theta \\ \left(\begin{matrix}\cos(\theta)\\ \sin(\theta)\end{matrix}\right. & \left.\begin{matrix}-r\sin(\theta)\\ r\cos(\theta)\end{matrix}\right)\end{matrix},\quad DG = \begin{matrix} \\ xy\\ x^2+y^2\end{matrix}\begin{matrix} x & y \\ \left(\begin{matrix} y\\ 2x\end{matrix}\right. & \left.\begin{matrix} x\\ 2y\end{matrix}\right)\end{matrix}$$

Since $H(r,\theta) = (r\cos(\theta), r\sin(\theta))$,

$$[DG\circ H](x,y) = \begin{pmatrix} r\sin(\theta) & r\cos(\theta)\\ 2r\cos(\theta) & 2r\sin(\theta)\end{pmatrix}.$$

Multiplying the DH gives:

$$[DG \circ H](x,y)DH(x,y) = \begin{pmatrix} r\sin(\theta) & r\cos(\theta) \\ 2r\cos(\theta) & 2r\sin(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}$$
$$= \boxed{\begin{pmatrix} 2r\sin(\theta)\cos(\theta) & r^2(\cos^2(\theta) - \sin^2(\theta)) \\ 2r & 0 \end{pmatrix}}$$

which is the same derivative we found by direct computation.

34.1: Let S be the surface defined by the intersection of the half cylinder $\{(x,y,z) : x \geq 0, x^2 + y^2 \leq 9\}$ with the plane $z = (1/2)y$. If the temperature at point (x,y,z) on this surface is x, find the heat flux from the surface.

Solution Use parameters r and θ (as in polar/cylindrical) coordinates to parameterize the surface. Then $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = (1/2)y = (1/2)r\sin(\theta)$. So the integral is

$$I = \int_S r\cos(\theta)\,\|dS\|.$$

To find $\|dS\|$, first look at dS:

$$\begin{aligned} dS &= \frac{\partial S}{\partial r} \times \frac{\partial S}{\partial \theta} \\ &= (\cos(\theta), \sin(\theta), (1/2)\sin(\theta)) \times r(-\sin(\theta), \cos(\theta), (1/2)\cos(\theta)) \\ &= r((1/2)\sin(\theta)\cos(\theta) - (1/2)\sin(\theta)\cos(\theta), (1/2)\sin(\theta)(-\sin(\theta) - \cos(\theta)(1/2)\cos(\theta), \\ &\quad \cos(\theta)\cos(\theta) - \sin(\theta)(-\sin(\theta))) \\ &= r(0, -1/2, 1). \end{aligned}$$

So $\|dS\| = |r|\sqrt{0^2 + (-1/2)^2 + 1} = r\sqrt{5/4}$. Hence the integral is

$$I = \int_S r\cos(\theta) r\sqrt{5/4}\ dS$$

Here S has $r \in [0,3]$ (since 3 is the radius of the cylinder) and $\theta \in [-\tau/4, \tau/4]$ (since $x \geq 0$.) Since the integrand is continuous and (r,θ) are in a compact space, Fubini can be used to turn this into an iterate integral and give:

$$\begin{aligned} I &= \int_{r=0}^{3} \int_{\theta \in [-\tau/4, \tau/4]} r^2 \cos(\theta)\sqrt{5/4}\ d\theta\ dr \\ &= \sqrt{5/4} \int_{r=0}^{3} r^2 \sin(\theta)|_{\theta=-\tau/4}^{\tau/4}\ \ dr \\ &= 2\sqrt{5/4} \int_{r=0} 32r^2\ dr \\ &= 2\sqrt{5/4}r^3/3|_0^3 \\ &= 9\sqrt{5} \approx \boxed{20.12}. \end{aligned}$$

35.1: What is the curvature of a circle of radius 2?

Solution The curvature of a circle is just the multiplicative inverse of the radius, so $1/2 = \boxed{0.5000}$.

37.1: Show the Divergence Theorem in 2D using Stokes' Theorem.

Solution Solution not included.

37.2: Show the Divergence Theorem in 3D using Stokes' Theorem.

Solution Solution not included.